Electromagnetism VIII: Synthesis

Electromagnetism in matter is covered in chapters 10 and 11 of Purcell, or in sections 6.4 and 7.5 of Wang and Ricardo, volume 2. All other problems combine ideas covered in previous problem sets. For more on dielectrics, see chapters II-10 and II-11 of the Feynman lectures. Electromagnetism in matter is covered in greater detail in chapters 4, 6, and 9 of Griffiths, and chapters I-31 and II-32 through II-37 of the Feynman lectures. There is a total of **81** points.

1 Dielectrics

Idea 1

When a dielectric is placed in an electric field, dipoles inside align with the field, reducing the field value. For the simplest, most symmetrical situations, the field is simply reduced by a factor of the dielectric constant $\kappa = \epsilon/\epsilon_0$. Hence a capacitor filled with dielectric has its capacitance enhanced by κ.

The simple fact above, along with physical intuition, will be enough for most problems. However, it's also sometimes useful to think about what's going on inside a dielectric.

Idea 2

Microscopically, a dielectric carries a polarization $\mathbf{P}$ with units of electric dipole moment density, describing the net effect of its dipoles. This results in a "bound" charge

$$\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$$

within the dielectric, as well a surface bound charge

$$\sigma_{\text{bound}} = \mathbf{P} \cdot \hat{\mathbf{n}}.$$

In a dielectric, the polarization is related to the total electric field by

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \epsilon = \epsilon_0(1 + \chi_e)$$

where χ_e is the electric susceptibility. The tricky thing about using $\mathbf{P}$ is that it depends on the *total* electric field, including the electric field produced by the polarization itself; that is, in general we have to solve for $\mathbf{P}$ in terms of itself.

Example 1

A point charge q is inside a dielectric κ. Find the electric field and charge density.

Solution

By idea 1, the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon r^2}\hat{\mathbf{r}}.$$

The dielectric simply shields the field by a factor of κ. To find the charge density, note that

$$\mathbf{P} = \frac{q}{4\pi r^2} \frac{\epsilon_0 \chi_e}{\epsilon} \hat{\mathbf{r}}.$$

The divergence of $\mathbf{P}$ is zero everywhere except for the origin, where negative bound charge piles up to cancel some of the charge q. (The compensating positive charge is at infinity, or the outer surface of the dielectric if it is finite.) The total charge at the origin is

$$q = q - q_{\text{bound}} = q\left(1 - \frac{\epsilon_0 \chi_e}{\epsilon}\right) = q\left(1 - \frac{\chi_e}{1+\chi_e}\right) = \frac{q}{\kappa}$$

which is consistent with Gauss's law for $\mathbf{E}$.

Example 2

A dielectric sphere of radius R and dielectric constant κ is placed in a field $\mathbf{E}_0$, and as a result develops a uniform polarization $\mathbf{P}$. Find $\mathbf{P}$ and the field everywhere.

Solution

First let's compute the field due to the sphere. The uniform polarization is equivalent to having two uniformly charged balls of total charge $\pm Q$ displaced by $\mathbf{d}$ so that $Q\mathbf{d} = (4\pi R^3/3)\mathbf{P}$. By the shell theorem, the field inside is uniform,

$$\mathbf{E}_p = -\frac{\mathbf{P}}{3\epsilon_0},$$

and the field outside is exactly a dipole field. Now we compute the magnitude of $\mathbf{P}$. The subtlety is that the atoms in the sphere see not to the applied field $\mathbf{E}_0$, but the total field $\mathbf{E}$,

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}, \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_p.$$

Solving the system, we find

$$\mathbf{E} = \frac{3}{\kappa+2}\mathbf{E}_0, \quad \mathbf{P} = 3\,\frac{\kappa-1}{\kappa+2}\,\epsilon_0 \mathbf{E}_0.$$

The polarizability α of each atom is defined as the dipole moment per applied field,

$$\mathbf{p} = \alpha \mathbf{E}_0$$

so we have shown above that

$$\alpha = \frac{3\epsilon_0}{n}\frac{\kappa-1}{\kappa+2}$$

where n is the number density of atoms. This is the Clausius–Mossotti formula; it relates the macroscopically measurable parameter κ to the microscopic parameter α.

[2] **Problem 1** (Purcell 10.10)**.** Assume that the uniform field $\mathbf{E}_0$ that causes the electric field in example 2 is produced by large capacitor plates very far away. The field lines tangent to the sphere hit each of the distant capacitor plates in a circle of radius r. Find r in terms of R and κ.

Solution. The field lines are tangent at the widest part of the sphere. Consider a Gaussian surface which is bounded by a distant capacitor plate, a horizontal slice through the middle of the sphere, and all of these field lines. Using the results of example 2, the charge contained inside is

$$Q = \pi R^2 \, \frac{3(\kappa - 1)}{\kappa + 2} \, \epsilon_0 E_0 - \pi r^2 \epsilon E_0$$

where the first term is from cutting the polarized sphere. The flux through this surface is

$$\Phi = \frac{3}{\kappa + 2} \, \pi R^2 E_0.$$

Applying Gauss's law, we have

$$r = \sqrt{\frac{3\kappa}{\kappa + 2}} \, R.$$

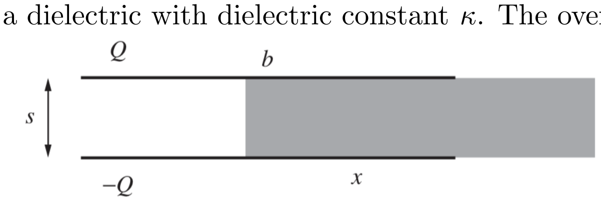
[2] **Problem 2** (Purcell 10.38)**.** Using a similar method to example 2, consider an infinite cylindrical rod of radius R with a fixed, uniform polarization $\mathbf{P}$, where $\mathbf{P}$ may have any orientation. (Don't worry about where $\mathbf{P}$ comes from; just assume it's "frozen into" the material. That is, the material is "ferroelectric".) Qualitatively describe the electric field everywhere.

Solution. First, note that the component of $\mathbf{P}$ parallel to the axis of symmetry of the cylinder doesn't do anything, because the cylinder is infinite. (You can think of the effect as moving charge from infinity in one direction to infinity in the other direction.) Thus, without loss of generality we can take $\mathbf{P}$ to be orthogonal to the axis of symmetry.

Now, the result is equivalent to having two uniform cylinders of total linear charge density λ separated by $\mathbf{d}$ where $\lambda \mathbf{d} = (\pi R^2)\mathbf{P}$. This is equivalent to two lines of charge density $\pm\lambda$ separated by $d = d'$ where d' is the length of the projection of $\mathbf{d}$ into the plane perpendicular to the axis of the lines. So outside, it's the field of two nearby, parallel lines of opposite charge.

Inside a cylinder with charge density ρ, an easy application of Gauss's law tells us that the field is $\mathbf{E} = \frac{\rho \mathbf{r}}{2\epsilon_0}$. Thus, combining the two cylinders, the field inside is $-\frac{\rho \mathbf{s}}{2\epsilon_0} = -\frac{\mathbf{P}}{2\epsilon_0}$, i.e. a constant.

[3] **Problem 3** (Purcell 10.2)**.** A rectangular capacitor with side lengths a and b has separation $s \ll a, b$. It is partially filled with a dielectric with dielectric constant κ. The overlap distance is x.



The capacitor is isolated and has constant charge Q.

(a) What is the energy stored in the system?

(b) Using the result of part (a), what is the force on the dielectric? Which direction does it point?

(c) Is your answer to part (b) affected by the presence of fringe fields near the interface?

Solution. (a) The system consists of two capacitors in parallel, with capacitances $C_1 = \epsilon_0(b - x)a/s$ and $C_2 = \kappa\epsilon_0 xa/s$. Thus,

$$C = \epsilon_0(a/s)((\kappa - 1)x + b)$$

which gives

$$U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 a(b + (\kappa - 1)x)}.$$

(b) Note that

$$F = -\frac{dU}{dx} = \frac{Q^2 s(\kappa - 1)}{2\epsilon_0 a(b + (\kappa - 1)x)^2}.$$

The sign is positive, so it points in direction of increasing x, so the slab is pulled in.

(c) Fringe fields don't change the result of part (b). The presence of fringe fields does change the energy found in part (a), but this has essentially no effect on the *derivative* of the energy, because shifting the dielectric just shifts the fringe field over essentially unchanged.

Of course, from a force perspective, *all* of the force is due to the fringe fields, because those are the only fields with a horizontal component; this paper gives such a calculation. The fact that you can get the same answer, by using an energy-based derivation that doesn't depend on the fringe fields, or by a force-based derivation that relies entirely on the fringe fields, is just another example of conservation of energy giving us nontrivial information.

[3] **Problem 4** (Griffiths 4.28)**.** Two long coaxial cylindrical metal tubes of inner radius a and outer radius b stand vertically in a tank of dielectric oil, with susceptibility χ_e and mass density ρ. The inner one is maintained at potential V, and the outer one is grounded. To what height h does the oil rise in the space between the tubes?

Solution. The field in the region with no oil is $E = \frac{\lambda}{2\pi\epsilon_0 r}$, and with the oil is $E' = \frac{\lambda'}{2\pi\epsilon r}$ where λ' is the free charge density. Thus,

$$V = \frac{\lambda}{2\pi\epsilon_0}\log(b/a),$$

and equating with the oil part, we get that $\lambda' = \kappa\lambda$, as expected. Now, the total charge on this effective capacitor is

$$Q = \lambda' h + \lambda(\ell - h) = \lambda(\chi_e h + \ell),$$

so

$$C = \frac{Q}{V} = 2\pi\epsilon_0 \frac{\chi_e h + \ell}{\log(b/a)}.$$

We know the net force is $\frac{1}{2}V^2(dC/dh)$ (note that there is not a minus sign here because of the work done by the battery, as explained in a problem in **E2**). The gravitational force is $\rho\pi g h(b^2 - a^2)$, so equating and solving for h gives

$$h = \frac{\epsilon_0 \chi_e V^2}{\rho(b^2 - a^2)g\log(b/a)}.$$

[4] **Problem 5** (Cahn)**.** [A] The region $z < 0$ is filled with a dielectric κ. Find the force on a point charge q a distance d above the origin.

Solution. First, we note there is only bound charge at the plane $z = 0$, and there cannot be net charge within the dielectric. This follows from taking the divergence of both sides of $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, which gives

$$-\rho_b = \chi_e \rho.$$

Since all the charge in the dielectric is bound, this equation can only be satisfied if $\rho = \rho_b = 0$.

Now, letting $\mathbf{E}_0$ be the electric field due to the point charge alone, we have

$$\mathbf{E}|_{z=0^+} = \mathbf{E}_0 + \frac{\sigma_b}{2\epsilon_0}\hat{\mathbf{z}}, \quad \mathbf{E}_{z=0^-} = \mathbf{E}_0 - \frac{\sigma_b}{2\epsilon_0}\hat{\mathbf{z}}.$$

Moreover, by the definition of χ_e we know that just inside the material,

$$\mathbf{P} = \epsilon_0 \chi_e \left(\mathbf{E}_0 - \frac{\sigma_b}{2\epsilon_0} \hat{\mathbf{z}} \right).$$

Taking the dot product of both sides with $\hat{\mathbf{z}}$ we have

$$\sigma_b = \epsilon_0 \chi_e \left(\mathbf{E}_0 \cdot \hat{\mathbf{z}} - \frac{\sigma_b}{2\epsilon_0} \right)$$

which rearranges to

$$\sigma_b = \frac{\chi_e}{\chi_e + 2} 2\epsilon_0 \mathbf{E}_0 \cdot \hat{\mathbf{z}} = \frac{\kappa - 1}{\kappa + 1} 2\epsilon_0 \mathbf{E}_0 \cdot \hat{\mathbf{z}}.$$

On the other hand, in the case of a perfectly conducting plane, we had $\mathbf{E}_{z=0^-} = 0$, giving

$$\sigma = 2\epsilon_0 \mathbf{E}_0 \cdot \hat{\mathbf{z}}.$$

That is, the charge density for the dielectric is identical except for a constant of $(\kappa - 1)/(\kappa + 1)$. Hence this bound charge is equivalent to an image charge, and the force is

$$F = \frac{q^2}{16\pi\epsilon_0 d^2} \frac{\kappa - 1}{\kappa + 1}$$

directed towards the dielectric.

2 Magnetic Materials

Idea 3

When a magnetic material is placed in an magnetic field, dipoles inside align with the field (for a paramagnet) or against the field (for a diamagnet). That is, both dielectrics and diamagnets reduce the applied field within them (the internal fields of electric and magnetic dipoles are opposite).

Idea 4

The configuration of a magnetic material is described by its magnetization $\mathbf{M}$, which has units of dipole moment per unit volume. This results in bound current density

$$\mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M}$$

as well as a surface bound current density

$$\mathbf{K}_{\text{bound}} = \mathbf{M} \times \hat{\mathbf{n}}.$$

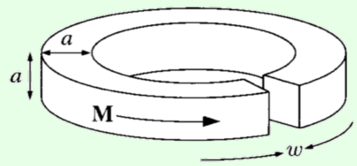
In a magnetic material, the magnetization obeys

$$\mathbf{M} = \frac{1}{\mu_0} \frac{\chi_m}{1 + \chi_m} \mathbf{B}, \quad \mu = \mu_0 (1 + \chi_m)$$

where χ_m is the magnetic susceptibility. (Note that χ_m is not defined the same way as χ_e.) Diamagnets have $\chi_m < 0$ and paramagnets have $\chi_m > 0$. Most common materials are only weakly magnetic, with $\mu \approx \mu_0$. Exceptions include superconductors, which are perfect diamagnets with $\chi_m = -1$ and hence $\mu = 0$, and ferromagnets, which have a frozen-in magnetization even when there's no external field, and hence no meaningful value of μ at all.

Example 3: Griffiths 6.10

An iron rod of length L and square cross section of side a is given a uniform longitudinal magnetization $\mathbf{M}$ and then bent into a circle with a narrow gap of width w.



Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$.

Solution

First, suppose there was no gap. Before the iron rod was bent, its uniform magnetization $\mathbf{M}$ corresponded to a bound current density $\mathbf{K}_{\text{bound}} = \mathbf{M}$ everywhere along its surface, directed circumferentially. Since $a \ll L$, this remains approximately true after bending the rod. (A small volume bound current density $\mathbf{J}_{\text{bound}}$ appears, but we neglect this.)

Therefore, the current density is the same as that of a toroidal solenoid with current I and n turns per length, where $M = In$. The field inside is therefore $\mu_0 M$, directed in the $\hat{\boldsymbol{\theta}}$ direction and zero everywhere outside the rod.

Now let's account for the gap. Adding the gap is equivalent to superposing an opposite magnetization at the gap. Since $w \ll a$, we can treat it as an $a \times a$ square current loop, with current $I = mw$. By the Biot–Savart law, the field due to such a loop at the center is

$$B_{\text{loop}} = \frac{2\sqrt{2}\mu_0 M w}{\pi a}.$$

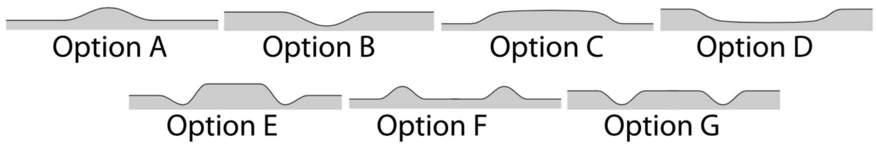
Combining the two gives a total field of

$$B = \mu_0 M \left(1 - \frac{2\sqrt{2}w}{\pi a}\right).$$

Since magnetization can be a bit mathematically nasty, you'll rarely be asked to find explicit fields, as in the above example. What's more important is the conceptual understanding.

[3] **Problem 6** (IPhO 2012). Water is a diamagnetic substance. A powerful cylindrical magnet with field B is placed below the water surface.

(a) Which of the following shows the resulting shape of the water surface?



The magnet is roughly 2/3 as wide as each of these sketches.

(b) Let ρ be the density of the water. If the maximum change in height of the water surface is h, find an approximate expression for $\mu - \mu_0$.

For a closely related problem, see EuPhO 2018, problem 2. Here, the pressure change due to the interaction of water with an extreme magnetic field causes water to boil. As covered in **T3**, this happens roughly when the water pressure hits zero, since the vapor pressure of water at room temperature is small.

Solution. (a) The potential energy of a dipole is $U = -\mathbf{m} \cdot \mathbf{B}$. In a diamagnet, $\mathbf{m}$ and $\mathbf{B}$ are antiparallel, so the potential energy of the water is positive where the magnetic field is strong. The water is thus repelled from regions of higher magnetic field, leading to option D.

(b) The surface of the water is an equipotential, so when the water surface dips, the higher magnetic dipole interaction energy must be exactly compensated by lower gravitational potential energy. Per volume, the former is

$$MB \approx \frac{B^2 \chi_m}{\mu_0} = \frac{B^2}{\mu_0^2}(\mu - \mu_0)$$

while the latter is $\rho g h$. Equating the two gives

$$\mu - \mu_0 \approx \frac{\mu_0^2 \rho g h}{B^2}.$$

However, there's a subtlety here. Our derivation of the potential energy in **E4** assumed that $\mathbf{m}$ had constant magnitude, but that's not true for substance like water. In the absence of any field, water has *no* magnetic dipole moment at all. Instead, its magnetic dipole moment grows linearly with the applied field. That means that the energy of a dipole is not just $-\mathbf{m} \cdot \mathbf{B}$, but rather

$$U = -\int \mathbf{m} \cdot d\mathbf{B} = -\frac{\mathbf{m} \cdot \mathbf{B}}{2}$$

where the 1/2 comes from taking the integral of a linear function. This factor of 2 means the true final answer is

$$\mu - \mu_0 \approx \frac{2\mu_0^2 \rho g h}{B^2}.$$

[3] **Problem 7.** In **E4** we covered USAPhO 2015, problem B2, which shows that the fields inside magnets differ, depending on whether they are made of "Ampere" or "Gilbert" dipoles. For example, consider a long, thin cylindrical magnet magnetized along its axis. In the Ampere model, the internal field $\mathbf{B}_0$ points along $\mathbf{M}$, while in the Gilbert model the internal field is approximately zero. We can try to distinguish between the models by drilling a hole into the magnet, and measuring the field inside the hole.

(a) Suppose we drill a long, thin cylindrical hole inside the magnet, parallel to its axis. Show that the field in this hole is the same in both the Ampere and Gilbert models.

(b) Suppose we drill a short, flat cylindrical hole inside the magnet. Show that the field in this hole is the same in both the Ampere and Gilbert models.

(c) Is there *any* way to tell the two models apart, by drilling holes in magnets and measuring the field inside the hole? What if you used different magnet shapes?

Solution. (a) The trick to answering this part and the next is that drilling the hole is like superposing another magnet with opposite magnetization. If we make a long thin cavity with the same shape as the magnet itself, then in the Ampere model the field is $\mathbf{B}_0 - \mathbf{B}_0 = 0$, while in the Gilbert model the field is $0 - 0 = 0$.

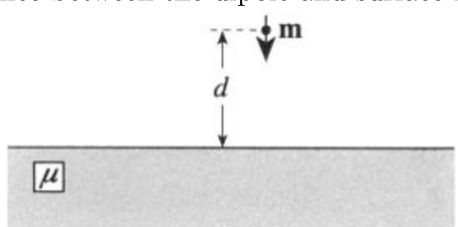
(b) In this case, in the Ampere model the field is about $\mathbf{B}_0 - 0$, while in the Gilbert model the field is about $0 - (-\mathbf{B}_0)$, so the two coincide again.

(c) No, the fields will always coincide. The only difference between an Ampere dipole and a Gilbert dipole is its internal field; far from such dipoles the fields are identical. If you drill a hole, then inside the hole, your detector will be far from all the dipoles in the material (which are atomic-sized), so it won't be able to tell the difference. That's why the Gilbert dipole description works so well.

[3] **Problem 8.** EFPhO 2004, problem 6. An elegant, tricky problem on permanent magnets.

Solution. See the official solutions here.

[4] **Problem 9** (Cahn)**.** [A] A small dipole $\mathbf{m}$ in vacuum points towards the plane surface of a medium with permeability μ. The distance between the dipole and surface is d.



Find the force acting on the dipole.

Solution. We'll parallel the solution to problem 5 as closely as possible. First, by the same reasoning as in that problem, there is only bound current at the plane $z = 0$.

Now, letting $\mathbf{B}_0$ be the magnetic field due to the dipole alone, we have

$$\mathbf{B}|_{z=0^+} = \mathbf{B}_0 + \frac{\mu_0}{2}\mathbf{K} \times \hat{\mathbf{z}}, \quad \mathbf{B}|_{z=0^-} = \mathbf{B}_0 - \frac{\mu_0}{2}\mathbf{K} \times \hat{\mathbf{z}}$$

by Ampere's law. Therefore, by the definition of χ_m, we know that just inside the material,

$$\mu_0 \mathbf{M} = \frac{\chi_m}{1+\chi_m}\left(\mathbf{B}_0 - \frac{\mu_0}{2}\mathbf{K} \times \hat{\mathbf{z}}\right).$$

Now we take the cross product of both sides with $\hat{\mathbf{z}}$. We know that $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{z}}$, and furthermore that taking the cross product of $\mathbf{K}$ with $\hat{\mathbf{z}}$ simply rotates it by 90° in the xy plane; hence $\mathbf{K} \times \hat{\mathbf{z}} \times \hat{\mathbf{z}} = -\mathbf{K}$. Then we have

$$\mu_0 \mathbf{K} = \frac{\chi_m}{1+\chi_m}\left(\mathbf{B}_0 \times \hat{\mathbf{z}} + \frac{\mu_0}{2}\mathbf{K}\right).$$

Upon solving for $\mathbf{K}$, we have

$$\frac{\mu_0}{2}\mathbf{K} = \frac{\chi_m}{\chi_m + 2}(\mathbf{B}_0 \times \hat{\mathbf{z}}) = \frac{\mu_r - 1}{\mu_r + 1}(\mathbf{B}_0 \times \hat{\mathbf{z}}).$$

On the other hand, in the case of a superconductor, we have $\mathbf{B}|_{z=0^-} = 0$, giving

$$\frac{\mu_0}{2}\mathbf{K} = -\mathbf{B}_0 \times \hat{\mathbf{z}}.$$

In this case, we know there is an image dipole of magnitude m directed opposite to the real dipole, and by the result of problem 12, the force between the dipoles is

$$F = \frac{3\mu_0}{2\pi}\frac{m^2}{(2d)^4}.$$

Note that unlike the case of a dielectric, this force is repulsive. For general μ_r, the surface current density is multiplied by a factor of $(\mu_r - 1)/(\mu_r + 1)$, so we have an image dipole at the same location but with smaller magnitude. The repulsive force is hence

$$F = \frac{3\mu_0}{32\pi}\frac{m^2}{d^4}\frac{\mu_r - 1}{\mu_r + 1}.$$

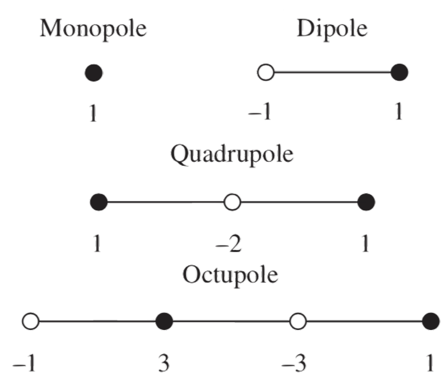
[5] **Problem 10.** Physics Cup 2012, problem 2.

Solution. See the official solutions here.

3 Multipoles

In this section, we explore some of the physics of dipoles and higher multipoles.

[3] **Problem 11** (Purcell 10.27)**.** Two monopoles of opposite sign form a dipole, two dipoles of opposite sign for a quadrupole, and so on. Hence we can construct arbitrarily high multipoles using the rows of Pascal's triangle.



The field of a dipole falls as $1/r^3$, a quadrupole as $1/r^4$, and an octupole as $1/r^5$.

(a) To warm up, verify explicitly that the quadrupole field along the axis of the quadrupole starts at $1/r^4$, i.e. that all lower terms cancel.

(b) Prove that this cancellation occurs for general multipoles along their axis.

(c) [A] The magnitude and orientation of a dipole is specified by a vector, with three components. How many numbers are necessary to specify the magnitude and orientation of a quadrupole? (The linear quadrupoles here are just a special case of a general quadrupole.) Try to generalize to arbitrary multipoles.

To learn how to decompose an arbitrary charge distribution into multipoles, see section 3.4 of Griffiths.

Solution. Since we're lazy we'll set the Coulomb constant $k = 1$, and the unit of charge also to 1, as well as the unit of distance spacing.

(a) See the solution to (b).

(b) A simple way to do this is to reason inductively. For example, an octupole field is nothing more than two quadrupoles whose leading terms cancel, so the leading field of an octupole has to be at least one power lower in r.

However, we will give an explicit proof. A 2^N-pole can be constructed from $N+1$ charges, with charge j placed at $x = -j$ with charge $(-1)^j\binom{N}{j}$. Then the field at point x is

$$E(x) = \sum_{j=0}^{N}(-1)^j\binom{N}{j}\frac{1}{(x+j)^2} = x^{-2}\sum_{j=0}^{N}\binom{N}{j}\sum_{k=0}^{\infty}\binom{-2}{k}(j/x)^k.$$

We see that this can be split into sums of the form $f(k) = \sum_{j=0}^{N}(-1)^j\binom{N}{j}j^k$, and the coefficient of x^{-2-k} is some nonzero multiple times $f(k)$. So it suffices to show that $f(k) = 0$ for all $k < N$, and $f(N) \neq 0$. This is an exercise in algebraic sums. The key idea is to define

$$g(k) = \sum_{j=0}^{N}(-1)^j\binom{N}{j}\binom{j}{k} = \sum_{j=k}^{N}(-1)^j\binom{N}{j}\binom{j}{k}.$$

We see that j^k can be written as a linear combination of $\binom{j}{0}, \ldots, \binom{j}{k}$, so it suffices to show that $g(k) = 0$ for all $k < N$, and that $g(N) \neq 0$. We see that

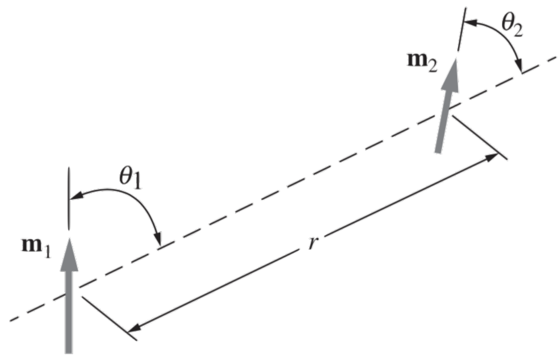
$$\begin{aligned} g(k) &= \sum_{j=k}^{N}(-1)^j\binom{N}{j}\binom{j}{k} \\ &= \sum_{j=k}^{N}(-1)^j\binom{N}{k}\binom{N-k}{j-k} \\ &= \binom{N}{k}\sum_{j=k}^{N}(-1)^j\binom{N-k}{j-k} \\ &= \binom{N}{k}(-1)^k\cdot\mathbf{1}_{k=N} \end{aligned}$$

where we used the fact that $\sum_{\ell=0}^{M}(-1)^\ell\binom{M}{\ell} = \mathbf{1}_{M=0}$ (here $\mathbf{1}_S$ is 1 if and only if S is true, and is 0 otherwise), which follows from the binomial theorem. This completes the proof.

(c) Let's think of a general quadrupole as a superposition of two dipoles in opposite directions. Then there are three things that determine a quadrupole: the strength of the quadrupole moment (i.e. the prefactor of the $1/r^4$ field), the orientation of the first dipole, and the direction the second dipole is displaced from it. This is $1 + 2 + 2 = 5$ total parameters.

Similarly, to specify an octupole, we do the same above, then specify the direction the second quadrupole is displaced, giving $5 + 2 = 7$ parameters. In general, a 2^N-pole has $2N + 1$ parameters.

[3] **Problem 12** (Purcell 11.23)**.** Two magnetic dipoles are arranged as shown.



Show that the associated potential energy is

$$U = \frac{\mu_0 m_1 m_2}{4\pi r^3} \left(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2 \right).$$

For what orientations is this potential energy maximized or minimized?

Solution. The magnetic field from a dipole pointing in the z direction is:

$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}}).$$

Let $\hat{\mathbf{n}}$ be the unit vector perpendicular to $\hat{\mathbf{r}}$ ($\theta = -\pi/2$). Then the field of $\mathbf{m}_1$ is

$$\mathbf{B}_{12} = \frac{\mu_0 m_1}{4\pi r^3}(2\cos\theta_1 \hat{\mathbf{r}} + \sin\theta_1 \hat{\mathbf{n}})$$

The potential of a dipole in a field is $U = -\mathbf{m} \cdot \mathbf{B}$. Note that $\mathbf{m}_2 = m_2 \cos\theta_2 \hat{\mathbf{r}} - m_2 \sin\theta_2 \hat{\mathbf{n}}$.

$$U = -\mathbf{m}_2 \cdot \mathbf{B}_{12} = \frac{\mu_0 m_1 m_2}{4\pi r^3}(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2)$$

as desired. To extremize this expression, we set the partial derivatives with respect to θ_1 and θ_2 equal to zero. The results are

$$\cos\theta_1 \sin\theta_2 = -2\sin\theta_1 \cos\theta_2, \quad \sin\theta_1 \cos\theta_2 = -2\cos\theta_1 \sin\theta_2$$

which implies that

$$\cos\theta_1 \sin\theta_2 = \sin\theta_1 \cos\theta_2 = 0.$$

This can only hold if

$$\cos\theta_1 = \cos\theta_2 = 0 \text{ or } \sin\theta_1 = \sin\theta_2 = 0.$$

The first option leads to local maxima or minima, where both the angles are $\pm\pi/2$, with maximum/minimum energy occurring when the dipoles are anti-aligned/aligned. The second option leads to the *global* maximum and minimum,

$$\text{maximum: } (0,0) \text{ or } (\pi,\pi), \quad \text{minimum: } (0,\pi) \text{ or } (\pi,0)$$

where the dipoles are anti-aligned/aligned, along the direction of the separation between them.

[2] **Problem 13** (Purcell 11.36)**.** Three magnetic compasses are placed at the corners of a horizontal equilateral triangle. As in any ordinary compass, each compass needle is a magnetic dipole constrained to rotate in a horizontal plane. The Earth's magnetic field has been shielded. What orientation will the compass needles eventually assume? Does your result also hold for regular N-gons?

Solution. We claim they point in the direction of the tangents to the circumcircle of the triangle. In this case, the field at any one corner due to the compasses at the other corners points in the tangential direction, so the compasses are all aligned with the local fields.

We can show this claim by symmetry. Consider the field at a given corner of the triangle. Flipping about the axis that passes through this corner and the midpoint of the opposite side negates the dipole moments at the other two corners, so it must negate the field. But physically, the rotation operation negates the tangential component of the field. So there must only be a tangential component, i.e. the field at this corner is purely tangential. This argument holds unchanged for regular N-gons.

[3] **Problem 14.** Some questions about forces between dipoles and other multipoles.

(a) Above, you've shown that the force between permanent magnetic dipoles falls off as $1/r^4$. How about two permanent electric dipoles?

(b) How about a permanent dipole and a permanent quadrupole?

(c) How about two permanent quadrupoles?

(d) Now consider an ion and a neutral atom. The electric field of the ion polarizes the atom; the field of that induced dipole then reacts on the ion. Show that the resulting force is attractive and falls as $1/r^5$.

Solution. (a) The basic form of the fields and forces is identical, so the answer is the same.

(b) The field of a quadrupole goes like $1/r^4$, so energy of the dipole goes like $U \sim mB \sim 1/r^4$. Thus, the interaction energy in this case goes like $1/r^4$, for a force of $1/r^5$.

(c) A single quadrupole is two dipoles with moments $\mathbf{m}$ and $-\mathbf{m}$ separated by $d\mathbf{r}$ where mdr is the quadrupole moment order. We have that the energy of the quadrupole is

$$\mathbf{m} \cdot \mathbf{B}(\mathbf{r} + d\mathbf{r}) - \mathbf{m} \cdot \mathbf{B} = \mathbf{m} \cdot ((\text{some sort of derivative of } \mathbf{B}) \cdot d\mathbf{r}).$$

The field of one quadrupole is $1/r^4$, so its derivative is $1/r^5$. Thus the energy of interaction goes like $1/r^5$, for a force of $1/r^6$.

(d) The field of the ion falls as $1/r^2$, so the dipole moment induced is $p \sim 1/r^2$. Furthermore, the dipole moment points along the field and hence the displacement between the ion and atom, indicating the force is attractive. The electric field from the dipole (and hence the force) goes as $p/r^3 \sim 1/r^5$. (You might wonder if the induced dipole then gives the ion itself a dipole moment. It does, but the resulting force is much weaker than the one we found here, between the induced dipole and the ion's overall charge.)

4 Electromagnetic Waves in Matter

In this section, you will work out some of the theory of electromagnetic waves in matter.

Idea 5

In the absence of any free charge or current, Maxwell's equations in matter are identical to Maxwell's equations in vacuum, except that ϵ_0 and μ_0 are related by ϵ and μ, so the waves propagate with speed $1/\sqrt{\epsilon\mu} = c/n$, with $E = (c/n)B$.

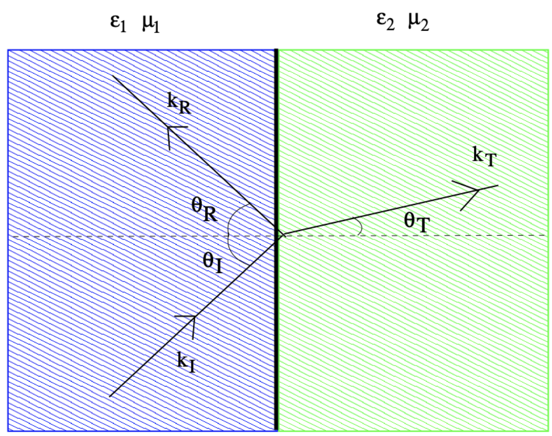
[5] **Problem 15.** Suppose the regions $x < 0$ and $x > 0$ are filled with material with permittivities ϵ_1 and ϵ_2, both with permeability μ_0. (This is typical; if you don't count permanent magnets, most objects have permeability about μ_0.) We send in an incident wave from the left with electric field

$$\mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)}.$$

The wave will be both transmitted and reflected at the interface, so the total electric field is

$$\mathbf{E} = \begin{cases} \mathbf{E}_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega_i t)} + \mathbf{E}_r e^{i(\mathbf{k}_r \cdot \mathbf{x} - \omega_r t)} & x < 0, \\ \mathbf{E}_t e^{i(\mathbf{k}_t \cdot \mathbf{x} - \omega_t t)} & x > 0. \end{cases}$$

The angles with the normal are θ_i, θ_r, and θ_t as shown.



(a) Argue that by continuity of the field at the boundary,

$$\omega_i = \omega_r = \omega_t.$$

(b) Suppose the y-axis is oriented so that $\mathbf{k}_i \cdot \hat{\mathbf{y}} = 0$. Argue that

$$\mathbf{k}_r \cdot \hat{\mathbf{y}} = \mathbf{k}_t \cdot \hat{\mathbf{y}} = 0, \quad \mathbf{k}_i \cdot \hat{\mathbf{z}} = \mathbf{k}_r \cdot \hat{\mathbf{z}} = \mathbf{k}_t \cdot \hat{\mathbf{z}}.$$

From these conditions, derive the laws of reflection and refraction,

$$\theta_i = \theta_r, \quad n_1 \sin\theta_i = n_2 \sin\theta_t.$$

Note that neither this part nor the previous part require Maxwell's equations; they hold for *all* kinds of waves as long as we define $n_i \propto 1/v_i$.

(c) Argue that at the boundary, $\mathbf{E}_\parallel$ and $B_\perp$ must be continuous in general. In this case, because both sides have the same permittivity μ_0, there is no bound current, so $\mathbf{B}_\parallel$ is also continuous.

(d) Now suppose the electric fields are polarized along the $\mathbf{y}$ axis, so $\mathbf{E}_i$, $\mathbf{E}_r$, and $\mathbf{E}_t$ are all parallel to the y-axis. Then continuity of $\mathbf{E}_\parallel$ gives

$$E_i + E_r = E_t.$$

Using continuity of $\mathbf{B}_\parallel$, show that

$$\frac{E_r}{E_i} = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}, \quad \frac{E_t}{E_i} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}.$$

These are the Fresnel equations for normal polarized light. (Hint: this is a bit messy, so you can warm up with the case $\theta_i = 0$.)

(e) If $n_1 > n_2$, then total internal reflection occurs when

$$\sin\theta_i > \frac{n_2}{n_1}$$

and the wave is totally reflected. Nonetheless, E_t is nonzero in this regime. To make sense of this, show that $\mathbf{k}_t \cdot \mathbf{x}$ is imaginary in this regime, indicating that the "transmitted" wave does not propagate in the region $x > 0$, but rather exponentially decays.

Solution. (a) Continuity of the field at the interface gives

$$E_i' e^{\omega_i t} + E_r' e^{\omega_r t} = E_t' e^{\omega_t t}.$$

If $\omega_i \neq \omega_t$, then the left side cannot be of the form $Ee^{\omega t}$, so $\omega_i = \omega_r$, and for the two sides to match up, we must have $\omega_i = \omega_r = \omega_t$.

Of course, the deeper reason behind this is just what we said in **M4** and will see again in **W1**. The differential equation the field obeys is linear, and has no explicit time dependence. Thus, it has solutions with uniform frequency everywhere.

(b) Again, look at the boundary and match parallel components of $\mathbf{E}$. By fixing z but varying y, we get an equation of the form

$$E_i' e^{\mathbf{k}_i \cdot y\hat{\mathbf{y}}} + E_r' e^{\mathbf{k}_r \cdot y\hat{\mathbf{y}}} = E_t' e^{\mathbf{k}_t \cdot y\hat{\mathbf{y}}}.$$

As before, for this to work, all the frequency factors of $\mathbf{k}_{\text{something}} \cdot \hat{\mathbf{y}}$ must be identical. We can do the same argument for $\hat{\mathbf{z}}$.

The first set gives that all the $\mathbf{k}$'s are in the same plane. The second gives that $k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$. We have $\omega/k = c/n$, so $k = n\omega/c$. Since all the ωs are the same, this equation then reads

$$n_1 \sin\theta_i = n_1 \sin\theta_r = n_2 \sin\theta_t,$$

which is exactly what we want.

(c) For $B_\perp$, consider a thin Gaussian pillbox that straddles the boundary. By Gauss's law for magnetism, the magnetic flux through it must be zero. In the limit of a very thin pillbox, this ensures the continuity of $B_\perp$.

For $\mathbf{E}_\parallel$, consider a thin Amperian loop that straddles the boundary, and consider $\oint \mathbf{E} \cdot d\mathbf{s}$. As the width of the loop goes to zero, the magnetic flux through it goes to zero, so this integral must be zero. Taking loops of various orientations, this ensures the continuity of $\mathbf{E}_\parallel$.

Note that $E_\perp$ and $\mathbf{B}_\parallel$ need not be continuous, because we can have surface charges and currents at the boundary. Since both sides have the same μ_0, there are no surface currents, so $\mathbf{B}_\parallel$ is continuous.

(d) The continuity of $\mathbf{B}_\perp$ gives

$$B_i \cos\theta_i - B_r \cos\theta_r = B_t \cos\theta_t.$$

Since $B = En/c$, this means

$$E_i n_1 \cos\theta_i - E_r n_1 \cos\theta_r = E_t n_2 \cos\theta_t.$$

Now with the continuity of $\mathbf{E}_\parallel$ ($E_i + E_r = E_t$), and $\theta_i = \theta_r$, we have

$$E_i n_1 \cos\theta_i - E_r n_1 \cos\theta_i = E_i n_2 \cos\theta_t + E_r n_2 \cos\theta_t$$

which yields

$$\frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}, \quad \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

as desired.

(e) As seen in part (b), we have

$$(k_i)_y = (k_t)_y, \quad (k_i)_z = (k_t)_z$$

The magnitudes of the wavenumbers are known:

$$k_t = \frac{\omega}{c} n_2, \quad k_i = \frac{\omega}{c} n_1, \quad k_t = k_i \frac{n_2}{n_1}$$

Then using the expression for the magnitude of $\mathbf{k}_t$:

$$(k_t)_x^2 = k_t^2 - (k_t)_y^2 - (k_t)_z^2 = \left(k_i \frac{n_2}{n_1}\right)^2 - (k_i)_y^2 - (k_i)_z^2$$

Since $\sin\theta_i$ represents $k_\parallel/k$, then for $\sin\theta_i = k_\parallel/k > n_2/n_1$:

$$(k_t)_x^2 = \left(k_i \frac{n_2}{n_1}\right)^2 - (k_i)_\parallel^2 < (k_i)_\parallel^2 - (k_i)_\parallel^2 = 0$$

from which we conclude that

$$(k_t)_x^2 < 0.$$

Thus the electric field will exponentially decay as it goes into the material.

[5] **Problem 16.** In most common materials, $\mu \approx \mu_0$ while ϵ depends on frequency. We'll investigate the origin of this frequency dependence below.

(a) Model an electron in an atom as a mass m with charge q attached to a spring, with natural frequency ω_0 and a damping force $-m\gamma\mathbf{v}$, in an electric field $\mathbf{E}_0 e^{-i\omega t}$. Write down the equation of motion for the electron.

(b) The atomic polarizability is $\mathbf{p} = \alpha\mathbf{E}$. Show that

$$\alpha = \frac{q^2/m}{-\omega^2 + \omega_0^2 - i\gamma\omega}.$$

(c) For a gas with small number density n, the Clausius–Mossotti formula reduces to

$$\epsilon = \epsilon_0 + n\alpha.$$

Therefore, the permittivity is generally a complex number. The wavevector and frequency are related by $k^2 = \mu\epsilon\omega^2$. Explain why the fact that ϵ is complex indicates that waves can be absorbed.

(d) What frequency maximizes the absorption rate of the electromagnetic waves? Roughly how many wavelengths does a wave propagate at this frequency before being absorbed?

(e) What frequency maximizes the speed of the electromagnetic waves, and what is that speed?

(f) Transparent objects such as glass can be modeled as having a very high resonant frequency, much higher than that of visible light. Does blue light or red light refract more when passing from air to glass?

The intuitive reason that these electrons can affect the propagation speed of light is because they emit secondary electromagnetic waves that are out of phase with the original wave; this "pushes" the phase of the composite wave forward or backward, affecting the phase velocity. A complete explanation can be found in chapter I.31 of the Feynman lectures.

Solution. (a) We have

$$m\ddot{\mathbf{r}} = -m\omega_0^2\mathbf{r} - m\gamma\mathbf{v} + q\mathbf{E}_0 e^{-i\omega t}.$$

(b) Suppose $\mathbf{r} = \mathbf{r}_0 e^{i\omega t}$ where $\mathbf{r}_0$ is potentially complex. Then, we see that $\mathbf{E}_0 \parallel \mathbf{r}_0$ and

$$-m\omega^2\mathbf{r} = -m\omega_0^2\mathbf{r} - m\gamma i\omega\mathbf{r} + q(E_0/r_0)\mathbf{r}.$$

Thus,

$$E_0/r_0 = \frac{m(\omega_0^2 - \omega^2 + i\gamma\omega)}{q}.$$

We have that $p = -r_0 q/E_0$ which yields the result.

(c) If ϵ is complex, then with $\mu \approx \mu_0$ and ω^2 being real, then $k^2 = \mu\epsilon\omega^2$ will also be complex. Thus with a complex wavevector $\mathbf{k}$, the field of $\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$ will exponentially decay since there will be a negative term from the complex wavevector $\mathbf{k}$.

(d) The absorption occurs exponentially from the complex component of kx. With $k = \omega\sqrt{\mu\epsilon} \approx \omega\sqrt{\mu_0\epsilon_0}(1 + \frac{n\alpha}{2\epsilon_0})$, the absorption rate is maximized when the complex component of k is maximized.

$$\beta \equiv \mathrm{Im}(k) = \mathrm{Im}\left(\frac{\omega n}{2c\epsilon_0}\alpha\right) = \frac{\omega n}{2c\epsilon_0}\frac{q^2/m}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}(\gamma\omega)$$

$$= \frac{q^2 \gamma n}{2mc\epsilon_0} \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

The maximum value of this occurs when

$$\frac{d\beta}{d\omega^2} \propto \left((\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2 - \omega^2(2(\omega^2 - \omega_0^2) + \gamma^2)\right) = 0$$

which simplifies to yield

$$\omega_0^4 - \omega^4 = 0.$$

Thus a frequency of $\omega = \omega_0$ will maximize the absorption rate of the electromagnetic wave. The electric field will have a factor of $e^{-\beta x}$, and at $\omega = \omega_0$, $\beta = \frac{q^2 n}{2\gamma m c \epsilon_0}$. The value of the real wavevector $\mathrm{Re}\, k$ will be close to (note that $\mathrm{Re}(\alpha) = 0$ at $\omega = \omega_0$):

$$\mathrm{Re}(k) = \frac{\omega_0}{c}\left(1 + \mathrm{Re}\left(\frac{n\alpha}{2\epsilon_0}\right)\right) = \frac{\omega_0}{c}$$

Then for the wave to fall off by a factor of e, the wave will need to travel a distance of $\frac{1}{\beta}$, which is $\frac{1}{\beta\lambda} = \frac{k}{2\pi\beta}$ wavelengths. Thus,

$$\frac{k}{2\pi\beta} = \frac{\omega_0 \gamma m \epsilon_0}{\pi q^2 n}$$

is the number of wavelengths it will travel before the amplitude gets reduced by a factor of e.

(e) The phase velocity is maximized when $\frac{\omega}{\mathrm{Re}\, k}$, or $\mathrm{Re}\, \frac{1}{\sqrt{\mu\epsilon}}$ is maximized.

$$v_p = \mathrm{Re}\, \frac{1}{\sqrt{\mu\epsilon}} \approx c\left(1 - \mathrm{Re}\, \frac{1}{2}\frac{n\alpha}{\epsilon_0}\right) = c + \frac{cq^2 n}{2m\epsilon_0} \frac{\omega^2 - \omega_0^2}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}$$

Differentiating with respect to ω^2 and finding where it's 0 yields:

$$(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2 - (\omega^2 - \omega_0^2)(2(\omega^2 - \omega_0^2) + \gamma^2) = 0$$

$$(\omega^2 - \omega_0^2)^2 = \omega_0^2 \gamma^2$$

$$\omega^2 = \omega_0^2 \pm \omega_0 \gamma$$

Looking at the original, the smaller solution yields the minimum velocity, and the larger solution yields the maximum velocity (which happens to be greater than c). The maximum phase velocity is

$$v_{\max} = c + \frac{cq^2 n}{2m\epsilon_0} \frac{\omega_0 \gamma}{(\omega_0\gamma)^2 + \gamma^2(\omega_0^2 + \omega_0\gamma)}$$

(f) From the previous part, we have

$$v_p = c - \frac{cq^2 n}{2m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}$$

and now we know that $\omega_0 \gg \omega$, so

$$v_p \approx c - \frac{cq^2 n}{2m\epsilon_0} \frac{\omega_0^2 - \omega^2}{\omega_0^4 - 2\omega_0^2\omega^2 + (\gamma\omega)^2} \approx c - c\frac{q^2 n}{2m\epsilon_0 \omega_0^4}(1 + \omega^2/\omega_0^2)$$

Thus increasing the frequency would decrease the speed of light and increasing the index of refraction, so blue light would refract more.

[5] **Problem 17.** IPhO 2002, problem 1. A neat application of electromagnetic waves in matter.

[5] **Problem 18.** APhO 2007, problem 2. A problem on an exotic negative index of refraction.

Remark

Above, we considered the response of a medium composed of atoms, obeying $p = \alpha E$. However, this relation is just an approximation, like Hooke's law. For larger electric fields, higher order terms are necessary,

$$p = \alpha E + \alpha' E^2 + \ldots$$

which lead to strange effects, studied in the field of nonlinear optics. For example, suppose we send in light of frequency ω. Then

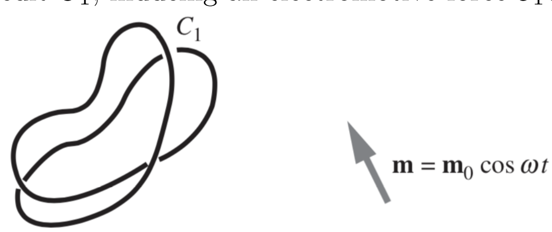
$$E^2 \propto \cos^2(\omega t) = \frac{1 + \cos(2\omega t)}{2}.$$

That means that a nonlinear medium can respond to light at frequency ω by oscillating, and hence emitting light, at frequency 2ω. This phenomenon is called frequency doubling, or second-harmonic generation, and converts red light to blue. Similarly, for a cubic nonlinearity, you can use trigonometric identities to show that frequency tripling can occur.

5 Electromagnetic Systems

In this section we'll consider problems that use everything we've covered, with a focus on technological applications and systems with multiple moving parts.

[2] **Problem 19** (Purcell 11.19)**.** A magnetic dipole $\mathbf{m}$ oscillates so that $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$. Some of its flux links the nearby circuit C_1, inducing an electromotive force $\mathcal{E}_1 \sin \omega t$.



If a current I_1 flowed in C_1, then the resulting field at the location of the dipole would be $\mathbf{B}_1$. Show that $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$. (Hint: recall the results involving mutual inductance in **E5**.)

Solution. Let there be a flux Φ_1 in circuit 1 and $\Phi_2 = \mathbf{B}_1 \cdot \mathbf{m}_0 / I_2$ in circuit 2. Then because $L_{12} = L_{21}$, as stated in **E5**, we have

$$\Phi_1/I_2 = \Phi_2/I_1 \implies \Phi_1 = \mathbf{B}_1 \cdot \mathbf{m}_0/I_2.$$

Then,

$$\mathcal{E}_1(t) = -d\Phi_1/dt = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0 \sin \omega t,$$

so $\mathcal{E}_1 = (\omega/I_1)\mathbf{B}_1 \cdot \mathbf{m}_0$.

[3] **Problem 20.** EFPhO 2007, problem 3. A problem on focusing particles with electric fields.

Solution. See the official solutions here.

[4] **Problem 21.** IPhO 2004, problem 3. A practical problem which also reviews damped/driven oscillations.

[4] **Problem 22.** EFPhO 2014, problem 1. A challenging problem about a complex nonlinear circuit.

Solution. See the official solutions here.

[5] **Problem 23.** Physics Cup 2020, problem 1.

Solution. See the official solutions here.