Mechanics VIII: Synthesis

Three-dimensional rotation is covered in chapter 7 of Kleppner and chapter 9 of Morin. For further discussion and examples, see chapter I-20 of the Feynman lectures, and this awesome video. There is a total of **101** points.

1 3D Rotation

In **M6**, we considered mostly two-dimensional rotation. Now we will tackle the full three-dimensional case, starting with the general description of rigid body motion.

Idea 1: Chasles' Theorem

The instantaneous velocity of a three-dimensional rigid body can always be decomposed in one of two ways. First, for any given point, it can be written in terms of a translational velocity plus a pure rotation about an axis going through that point. In practice, this point is almost always chosen to be the center of mass, giving the decomposition

$$\mathbf{v} = \mathbf{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}).$$

Alternatively, there always exists an axis so that the motion can be written as rotation about that axis, plus a translational velocity *parallel* to the axis, giving

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

where $\mathbf{v}_0$ and $\boldsymbol{\omega}$ are parallel. (This is known as a "screw" motion.) These two decompositions are analogous to the two we saw in **M5**, though in the three-dimensional case the first tends to be much more useful.

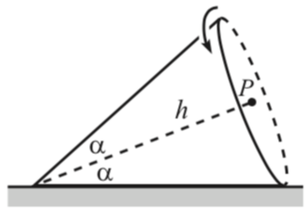
Example 1

You have a small globe, which is mounted so that it can spin on the polar axis and can be spun about a horizontal axis (so that the south pole can be on top). Give the globe a quick spin about the polar axis, and then, before it stops, give it another quick spin about the horizontal axis. Are there any points on the globe that are at rest?

Solution

The first spin gives the angular velocity a vertical component $\boldsymbol{\omega}_1$. The second spin gives the angular velocity an additional horizontal component $\boldsymbol{\omega}_2$. The globe now rotates about its center of mass with angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Precisely two points on the globe are stationary, namely the points that are parallel and antiparallel to this vector.

[2] **Problem 1** (Morin 9.3)**.** A cone rolls without slipping on a table; this means that all the points of the cone that touch the table are instantaneously stationary. The half-angle of the vertex is α, and the axis has length h.



Let the speed of the center of the base, point P in the figure, be v.

(a) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as pure rotation about some axis.

(b) Compute the angular velocity $\boldsymbol{\omega}$ by thinking of the motion as translation of P, plus rotation about an axis passing through P.

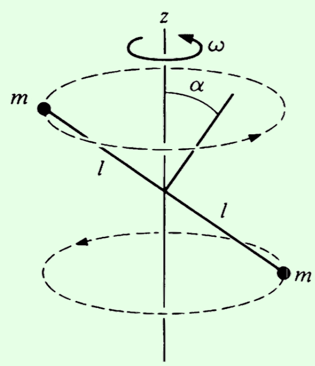
(c) The apex of the cone is fixed, and the cone continues to rotate. As this motion goes on, the angular velocity vector rotates uniformly, keeping a constant magnitude. Find the angular velocity $\boldsymbol{\Omega}$ of the angular velocity vector.

In the precession problems below, it's important to keep track of the difference between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$.

Most of our statements about rotational dynamics from **M6** remain true. The main new aspect is that angular momentum is not necessarily parallel to angular velocity.

Example 2: KK Example 7.4

Consider a rigid body consisting of two particles of mass m connected by a massless rod of length 2ℓ, rotating about the z-axis with angular velocity ω as shown.



Find the angular momentum of the system.

Solution

We simply add $\mathbf{r} \times \mathbf{p}$ for both masses. Let the rod lie in the xz plane at this moment. Then for the top left mass,

$$\mathbf{r} = -\ell \cos\alpha \, \hat{\mathbf{x}} + \ell \sin\alpha \, \hat{\mathbf{z}}.$$

The momentum is

$$\mathbf{p} = m\mathbf{v} = m\boldsymbol{\omega} \times \mathbf{r} = -m\omega\ell \cos\alpha \, \hat{\mathbf{y}}.$$

Then the angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\omega\ell^2 \cos\alpha \left(\sin\alpha \, \hat{\mathbf{x}} + \cos\alpha \, \hat{\mathbf{z}}\right).$$

The other mass has the opposite $\mathbf{r}$ and $\mathbf{p}$ and hence the same $\mathbf{L}$, so the total angular momentum is

$$\mathbf{L} = 2m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}})\,.$$

It is directed perpendicular to the rod, and in particular, it isn't parallel to the angular velocity!

Here is another way to derive the same result. We can decompose the angular velocity vector into a component along the rod, and a component perpendicular to the rod. The former contributes no angular momentum, because rotating about the rod's axis doesn't move the masses. The latter contributes all the angular momentum. So the angular momentum is

$$L = I_\perp \omega_\perp = (2m\ell^2)(\omega\cos\alpha)$$

directed perpendicular to the rod, which is what we just saw explicitly.

We can summarize the lessons drawn from this example as follows.

Idea 2

For a three-dimensional object, $\mathbf{L}$ is not necessarily parallel to $\boldsymbol{\omega}$. In general, for pure rotation about an axis passing through the origin, we have $\mathbf{L} = I\boldsymbol{\omega}$ where I is a 3×3 matrix called the "moment of inertia tensor about the origin". In components, this means that

$$L_i = \sum_j I_{ij}\omega_j.$$

While this is simple and general, the I_{ij} are a pain to calculate. You can learn more in the reading, but to my knowledge, no Olympiad problem has ever required computing a general moment of inertia tensor.

For the purposes of Olympiad problems, there is a better way to think about the angular momentum. We use the second decomposition of idea 1, and think of the motion as translation plus rotation about the center of mass. If the object has an axis of symmetry, which it will in almost all Olympiad problems, then the angular velocity can then be decomposed into a component parallel to the axis, and perpendicular to the axis,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\parallel + \boldsymbol{\omega}_\perp.$$

The key is that, in such situations, the spin angular momentum has two pieces, which are each parallel to the corresponding piece of the angular velocity,

$$\mathbf{L}_\parallel = I_\parallel \boldsymbol{\omega}_\parallel, \quad \mathbf{L}_\perp = I_\perp \boldsymbol{\omega}_\perp$$

where $I_\parallel$ and $I_\perp$ are ordinary moments of inertia about the center of mass. For example, for a flat uniform disc, $I_\parallel = MR^2/2$ and $I_\perp = MR^2/4$.

The total angular momentum about the origin is then

$$\mathbf{L} = \mathbf{r}_{\mathrm{CM}} \times M\mathbf{v}_{\mathrm{CM}} + I_\parallel \boldsymbol{\omega}_\parallel + I_\perp \boldsymbol{\omega}_\perp$$

where the first term is from the motion of the center of mass, and the next two are from rotation about the center of mass. Note that this is exactly the same as what we saw in **M5**, except that the "spin" angular momentum is broken into two parts.

Idea 3

Sometimes it can be hard to visualize $\boldsymbol{\omega}$, so here are two tricks. First, if any two points on the object are stationary, then $\boldsymbol{\omega}$ must be parallel to the axis connecting the two points. Second, if the rotation is complicated, one can use rotating frames to simplify the analysis. If a body has angular velocity $\boldsymbol{\omega}_1$ in one frame, and that frame has an angular velocity $\boldsymbol{\omega}_2$ with respect to a second frame, then the body has angular velocity $\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ with respect to the second frame.

Idea 4

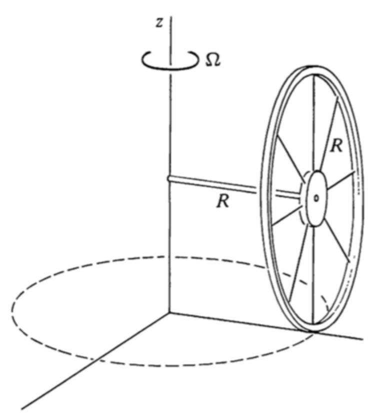
The rate of change of angular momentum is

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}$$

where the torque $\boldsymbol{\tau}$ is defined as in **M5**. The kinetic energy is

$$K = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I_{\parallel}\omega_{\parallel}^2 + \frac{1}{2}I_{\perp}\omega_{\perp}^2.$$

[3] **Problem 2** (KK 7.1)**.** A thin hoop of mass M and radius R rolls without slipping about the z-axis. It is supported by an axle of length R through its center, as shown.



The axle circles around the z-axis with angular speed Ω, so that the bottom point of the wheel traces out a circle of radius R. Let O be the pivot point of the rod, i.e. the point where the rod meets the z-axis.

(a) Find the instantaneous angular velocity $\boldsymbol{\omega}$ of the hoop.

(b) As the motion continues, the angular velocity vector rotates in a circle. Find the angular velocity of the angular velocity vector of the hoop.

(c) Find the instantaneous angular momentum $\mathbf{L}$ of the hoop, about the point O.

(d) Find the instantaneous torque on the hoop about the contact point with the ground, and verify that $\boldsymbol{\tau} = d\mathbf{L}/dt$.

The solution to this question will be the basis for all the rest, so ask if you have any questions!

Finally, we need just a few more facts to get going with dynamics problems.

Idea 5

In any dynamics problem, there are many choices you can make in the setup. For example, if you're using an inertial frame, you need to choose where the origin is; usually it's best to place it along the axis of symmetry if possible. You are also free to use a noninertial frame with acceleration $\mathbf{a}$. The only difference is that there will be a fictitious force $-M\mathbf{a}$ acting at the center of mass. For that reason, it's usually best to have the accelerating frame follow the center of mass, keeping it at its origin, so no new torques are introduced.

However, you should avoid *rotating* reference frames for dynamics problems. Not only will there be position-dependent Coriolis forces, but they'll add up and contribute a Coriolis *torque*, which is a pain to calculate, as you saw in **M6**. In general, rotating frames are only good for getting a handle on the kinematics, as mentioned in idea 3.

Example 3: KK Example 7.5

Calculate the magnitude of the torque on the rod in example 2.

Solution

We recall that the angular momentum was

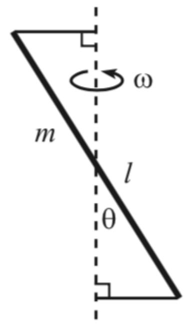
$$\mathbf{L} = 2m\omega\ell^2 \cos\alpha\,(\sin\alpha\,\hat{\mathbf{x}} + \cos\alpha\,\hat{\mathbf{z}}).$$

The rod as a whole rotates with angular velocity $\omega\hat{\mathbf{z}}$. In particular, the angular momentum vector rotates with this angular velocity as well; its horizontal component moves in a circle with angular velocity ω. Then

$$|\boldsymbol{\tau}| = \left|\frac{d\mathbf{L}}{dt}\right| = \omega L_x = 2m\omega^2\ell^2\,\cos\alpha\sin\alpha = m\omega^2\ell^2\sin(2\alpha).$$

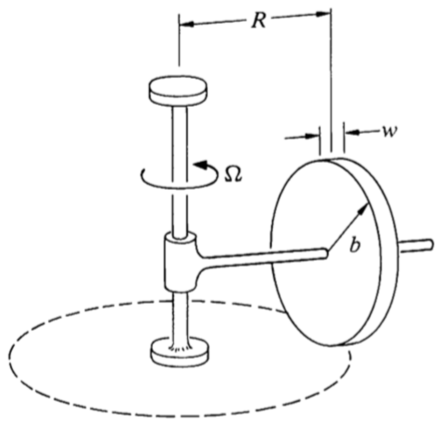
It might be surprising that there needs to be a torque given that $\boldsymbol{\omega}$ is constant, but that's just because $\mathbf{L}$ and $\boldsymbol{\omega}$ aren't necessarily parallel. Conversely, there can be situations where there is no torque, yet $\boldsymbol{\omega}$ changes over time.

[2] **Problem 3** (Morin 9.10). A stick of mass m and length ℓ spins with frequency ω around an axis in zero gravity, as shown.



The stick makes an angle θ with the axis and is kept in its motion by two strings that are perpendicular to the axis. Find the tension in the strings.

[2] **Problem 4** (KK 7.4)**.** In an old-fashioned rolling mill, grain is ground by a disk-shaped millstone which rolls in a circle on a flat surface driven by a vertical shaft. Because of the stone's angular momentum, the contact force with the surface can be greater than the weight of the wheel.



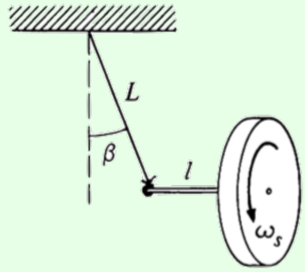
Assume the millstone is a uniform disk of mass M, radius b, and width w, and it rolls without slipping in a circle of radius R with angular velocity Ω. Find the contact force. Assume the millstone is closely fitted to the axle so that it cannot tip, and $w \ll R$. Neglect friction.

[2] **Problem 5** (Morin 9.29)**.** A uniform ball rolls without slipping on a table. It rolls onto a piece of paper, which you then slide around in an arbitrary (horizontal) manner. You may even give the paper abrupt, jerky motions, so that the ball slips with respect to it. After you allow the ball to come off the paper, it will eventually resume rolling without slipping on the table. Show that the final velocity equals the initial velocity. (Hint: this remarkably simple result is because of a conservation law. We saw a lower-dimensional version of this problem in **M5**.)

When a system such as a gyroscope is given a high angular momentum, it can exhibit precession. In this case, the angular velocity $\boldsymbol{\omega}$ precesses (i.e. rotates) with a small angular velocity $\boldsymbol{\Omega}$. Precession is a famously counterintuitive phenomenon, but it can be handled using just the same principles we've laid out above.

Example 4: KK 7.3

A gyroscope wheel is at one end of an axle of length ℓ. The other end of the axle is suspended from a string of length L.



The wheel is set into motion so that it executes slow, uniform precession in the horizontal plane. The wheel has mass M and moment of inertia I_0 about its center of mass, and turns with angular speed ω_s. Neglect the mass of the shaft and string. Find the angle β the string makes with the vertical, assuming β is very small.

Solution

Let T be the tension in the rope. The entire system precesses with associated angular velocity $\Omega\hat{\mathbf{z}}$. (That is, this is the angular velocity of the angular velocity and angular momentum vectors.) Since the center of mass does not accelerate vertically, and the center of mass moves in a horizontal circle,

$$T\cos\beta = Mg, \quad T\sin\beta = M\Omega^2(\ell + L\sin\beta).$$

We'll work to lowest possible order in β everywhere, which means approximating $\cos\beta \approx 1$ and ignoring the $L\sin\beta$ term, giving

$$T = Mg, \quad T\beta = M\Omega^2\ell.$$

Combining these equations gives the precession frequency

$$\Omega = \sqrt{\frac{g\beta}{\ell}}.$$

This is as far as we can go with forces alone.

Now we use $\boldsymbol{\tau} = d\mathbf{L}/dt$. First we need to determine the angular velocity of the wheel. Note that if we went in the rotating frame with angular velocity $\Omega\hat{\mathbf{z}}$, the wheel would just spin with angular speed ω_s in place. So the angular velocity vector in the original frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega_s\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is directed along the rod. The center of mass of the wheel moves with speed $(\ell + L\sin\beta)\Omega$ in a horizontal circle.

Since the precession is assumed to be slow, Ω is much smaller than ω_s. Thus, we can simply ignore contributions to the angular momentum proportional to Ω. That is, we can take

$$\boldsymbol{\omega} \approx \omega_s\hat{\mathbf{x}}$$

for the purposes of computing angular momentum, giving

$$\mathbf{L} \approx I_0 \omega_s \hat{\mathbf{x}}.$$

This rotates in a horizontal circle with angular speed Ω, so

$$|\boldsymbol{\tau}| \approx I_0 \Omega \omega_s.$$

The torques in the relevant direction come from gravity and the vertical component of the tension force, $|\boldsymbol{\tau}| = Mg\ell$. Equating these, we have

$$Mg\ell = I_0 \Omega \omega_s.$$

Plugging the result for Ω in above and solving for β gives

$$\beta = \frac{m^2 g \ell^3}{\omega^2 I_0^2}.$$

Note that assuming Ω was small kept the equations simple. This is what Kleppner calls the gyroscope approximation. It can be applied in some, but not all, of the below problems.

Remark

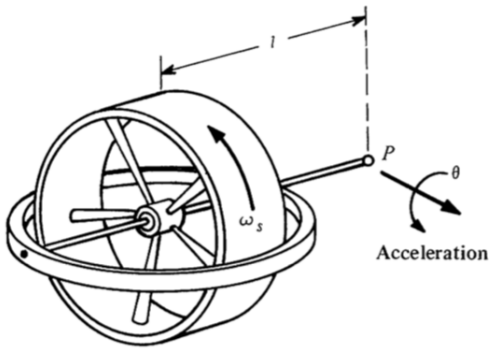
In most gyroscope problems, such as the one above, we assume the motion has reached a steady state, but you might wonder just how it gets started in the first place. For example, suppose we had the same setup as the previous problem, with the wheel spinning and the axle horizontal. For simplicity, let's get rid of the string and suppose the end of the axle is held at a fixed support. Now suppose the axle and wheel are released with no translational motion.

The following chain of events ensues:

- Of course, the axle starts to tip downward because of the weight of the wheel. (Rotational mechanics is counterintuitive, but not *that* counterintuitive!)
- This produces a downward component of angular momentum, which is balanced by the axle/wheel system twisting about its center of mass, rotating slightly about the z-axis.
- This twist tries to push the end of the axle out of the page, causing the support to exert a force on the axle pointing into the page. That force propagates down the axle as an internal shear stress, eventually causing the center of mass of the wheel to start moving into the page, starting the precession.
- In reality, the process overshoots and overcorrects, leading to oscillations called nutation on top of the precession. (For more details, see Note 2 of chapter 7 of Kleppner.)
- For a typical pivot, energy can be dissipated at the pivot point, but the angular momentum of the system stays roughly the same. Assuming this is the case, the oscillations will eventually damp away, leaving the uniform precession.

Notice that in this example, the initial angular momentum is perfectly horizontal. The final angular momentum includes an upward component due to the uniform precession, which implies that the axle must tilt slightly downward, by an angle of order $(\omega/\Omega)^2$. Therefore, if you want to set up uniform precession with the axle perfectly horizontal, as in the above example, you should point the axle slightly upward when releasing it from rest.

[2] **Problem 6** (KK 8.5). An "integrating gyro" can be used to measure the speed of a vehicle. Consider a gyroscope spinning at high speed ω_s. The gyroscope is attached to a vehicle by a universal pivot. If the vehicle accelerates in the direction perpendicular to the spin axis at rate a, then the gyroscope will precess about the acceleration axis, as shown.



The total angle of precession is θ. Show that if the vehicle starts from rest, its final speed is

$$v = \frac{I_s \omega_s}{M\ell}\theta$$

where $I_s\omega_s$ is the gyroscope's spin angular momentum, M is the total mass, and ℓ is the distance from the pivot to the center of mass.

[3] **Problem 7** (KK 7.5). When an automobile rounds a curve at high speed, the weight distribution on the wheels is changed. For sufficiently high speeds, the loading on the inside wheels goes to zero, at which point the car starts to roll over. This tendency can be avoided by mounting a large spinning flywheel on the car.

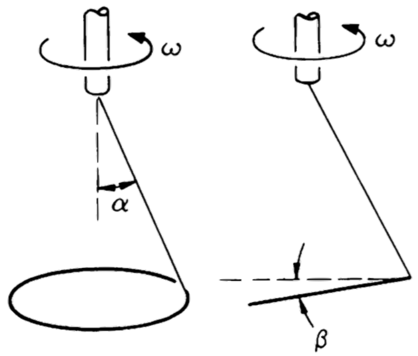
(a) In what direction should the flywheel be mounted, and what should be the sense of rotation, to help equalize the loading? (Check your method works for the car turning in either direction.)

(b) Show that for a disk-shaped flywheel of mass m and radius R, the requirement for equal loading is that the angular velocity ω of the flywheel is related to the velocity of the car v by

$$\omega = 2v\frac{ML}{mR^2}$$

where M is the total mass of the car and flywheel, and L is the height of their center of mass.

[3] **Problem 8** (KK 7.7). A thin hoop of mass M and radius R is suspended from a string through a point on the rim of the hoop. If the support is turned with high angular velocity ω, the hoop will spin as shown, with its plane nearly horizontal and its center nearly on the axis of the support.

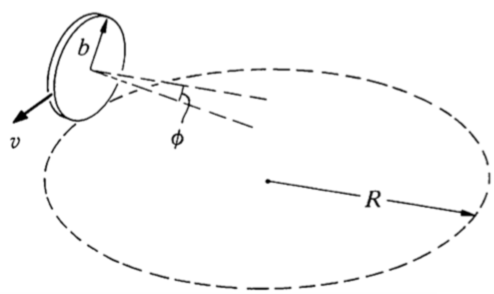


The string makes an angle α with the vertical.

(a) Find, approximately, the small angle β between the plane of the hoop and the horizontal.

(b) Find, approximately, the radius of the small circle traced out by the center of mass.

[4] **Problem 9** (KK 7.6, Morin 9.23)**.** With the right initial conditions, a coin on a table can roll in a circle.

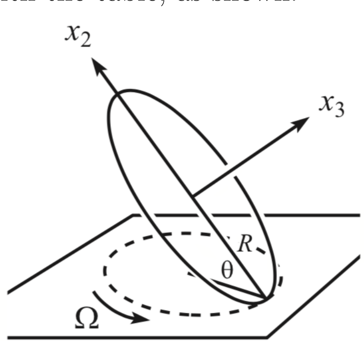


As shown, the coin leans inward, with its axis tilted to the horizontal by an angle ϕ. The radius of the coin is b, the radius of the circle it follows on the table is R, and its velocity is v.

(a) Assuming the coin rolls without slipping and $b \ll R$, show $\tan\phi = 3v^2/2gR$.

(b) No longer assuming $b \ll R$, show that the described motion is only possible if $R > (5/6)b\sin\phi$.

[4] **Problem 10** (Morin 9.24)**.** If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will gradually decrease, and eventually it will come to rest. Assume this process is slow, and consider the motion when the coin makes an angle θ with the table, as shown.



You may assume that the center of mass is essentially motionless. Let R be the radius of the coin, and let Ω be the frequency at which the contact point on the table traces out its circle. Assume the coin rolls without slipping.

(a) Show that the angular velocity of the coin is $\boldsymbol{\omega} = \Omega \sin\theta\, \hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ always points upward along the coin, directly away from the contact point.

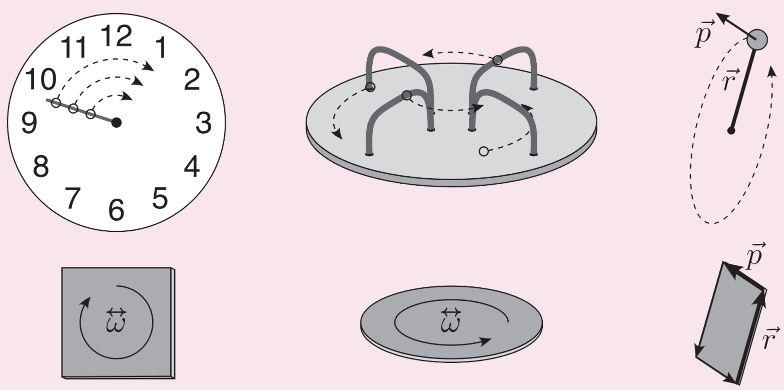
(b) Show that $\Omega = 2\sqrt{g/R\sin\theta}$.

(c) Show that the face on the coin appears to rotate, when viewed from above, with angular frequency $(1-\cos\theta)\Omega$.

Remark: Bivectors

Vector quantities defined by the cross product have some unusual properties. For example, under a spatial inversion, which flips the signs of $\mathbf{r}$ and $\mathbf{p}$, the sign of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ doesn't get flipped, so $\mathbf{L}$ transforms differently from other vectors. The same applies to the velocity $\boldsymbol{\omega}$ and magnetic field $\mathbf{B}$. All three of these quantities are "pseudovectors", not true vectors.

The underlying reason is that all of these quantities are fundamentally a different kind of mathematical object. They are really rank 2 differential forms, also called bivectors in three dimensions. While a vector is specified by an arrow with magnitude and direction, a bivector is specified by a planar tile with area and orientation. The following figure, taken from this paper, shows how it can be constructed visually from the cross product.



In three dimensions, we can always convert between bivectors and pseudovectors using the right-hand rule, so any calculation can be done with either form. Bivectors have the advantage of visually representing rotational quantities: the angular velocity bivector lies along an object's plane of rotation, while the magnetic field bivector lies along the plane in which it makes charged particles circularly orbit. However, it is easier to add vectors, both visually and mathematically, which also makes it easier to think about decomposing vectors into components. This advantage is so important in practice that I don't recommend using bivectors at all for three-dimensional problems.

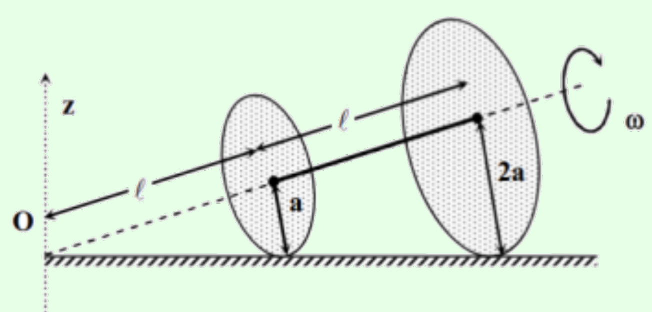
On the other hand, when you work in higher-dimensional spaces, the differential form perspective becomes indispensible. In general, in d dimensions the angular velocity has $\binom{d}{2}$ components, corresponding to the rotation rate in each independent plane.

- Of course, when $d = 1$ there is no such thing as rotation at all, while when $d = 2$ the angular velocity has one component, so we treat it as a scalar.
- When $d = 3$ the angular velocity has three components, so we treat it as a vector.
- When $d = 4$ the angular velocity has four components, so we can't even pretend it's a vector; we have to use the differential form description.

By the way, if you want to look into this material more, be sure to steer clear of "geometric algebra", which dominates the Google search results. Geometric algebra is a strange internet cult which recruits unsuspecting young people by telling them about bivectors, which are indeed cool. Once they have your attention, they'll claim that "mainstream" physics has hit a dead end because it refuses to go beyond vector notation, and then you'll spend years relearning all of physics in their wacky alternative notation. The truth is that physicists don't teach geometric algebra because it's not that useful when $d = 3$, while in higher dimensions we use tensor calculus and differential forms, which are much more powerful than either vectors or geometric algebra. More generally, if a physics movement has tons of internet presence but no actual textbooks or novel results, it's not worth paying attention to.

Example 5: IIT JEE 2016

Two thin circular discs, with radii a and $2a$, are connected by a rod of length $\ell = \sqrt{24}\,a$ through their centers. This rigid object rolls without slipping on a flat table.



The center of mass of the object rotates about the z-axis with an angular speed of Ω. The angular speed of the object about the axis of the rod is ϖ. How are Ω and ϖ related?

Solution

This is the most famous problem ever set on the IIT JEE (condensed for clarity), celebrated by generations of students for its difficulty. But it's also an example of how *not* to write a 3D rotation problem. Under the standard definition of angular velocity, none of the options provided in the question were correct, while the intended answer requires a nonstandard, arbitrary definition. You can find a detailed explanation of this here, by one of the former top scorers on the JEE, and I'll give a condensed explanation below.

First, let's figure out what's going on. The kinematics of this problem isn't any different from problem 1. Defining the x-axis to be horizontal in the figure above, the instantaneous angular velocity is $\boldsymbol{\omega} = \omega\,\hat{\mathbf{x}}$, while the "angular velocity of the angular velocity", describing the precession, is $\boldsymbol{\Omega} = (\omega/\sqrt{24})\,\hat{\mathbf{z}}$. The hard part is figuring out what the question writers meant by "the angular speed ϖ about the axis of the rod".

If we're only talking about the object's *instantaneous* motion, then the only possible answer is $\varpi = \boldsymbol{\omega} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector pointing along the rod. In that case we have $\Omega/\varpi = 5/24$, which wasn't an answer choice in the exam. On the other hand, if we are comparing the object's orientation at different times, then there isn't a unique answer. At a finite time later, the object will be in a different place, and computing a relative angle requires defining a convention for comparing orientations.

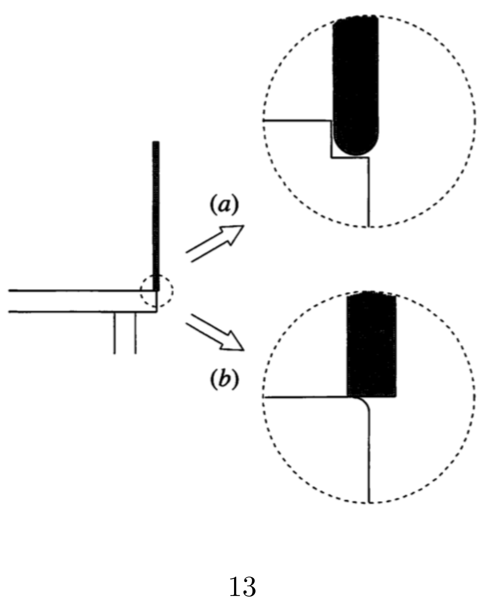
Here's what the problem authors meant. We work in the frame rotating with angular velocity $\boldsymbol{\Omega}$. In this frame, the system is spinning in place, with angular velocity $\boldsymbol{\omega} + \boldsymbol{\Omega}$ parallel to $\hat{\mathbf{n}}$. The definition of ϖ is $|\boldsymbol{\omega} + \boldsymbol{\Omega}|$, which gives $\Omega/\varpi = 1/5$, the intended answer.

Another way of saying this is that when we compare the orientation of the system at one moment to its orientation at another moment, we bring them to the same position by *rotating about the z-axis*, at which point they differ by a rotation about $\hat{\mathbf{n}}$. But this procedure is totally arbitrary, and not specified by the problem. To pose the problem properly, the writers could have either defined ϖ explicitly in the rotating frame mentioned above, or replaced it with a quantity with equivalent but unambiguous physical meaning, such as the interval between times a given point on the rim of a disc touches the ground. Fortunately, you'll almost never see ambiguous problems like this on Olympiads.

2 Composite Rotation

These are rotational dynamics problems like the ones you saw in **M5**, but more complex.

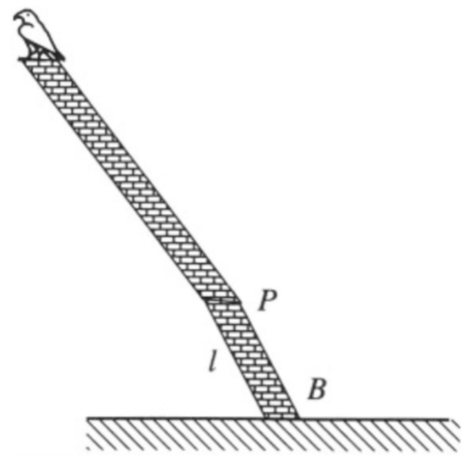
[3] **Problem 11** (PPP 60)**.** A uniform thin rod is placed with one end on the edge of a table in a nearly vertical position and then released from rest. Find the angle it makes with the vertical at the moment it loses contact with the table. Investigate the following two extreme cases.



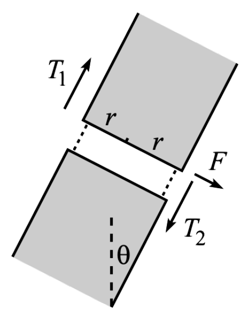
(a) The edge of the table is smooth (friction is negligible) but has a small, singe-step groove.

(b) The edge of the table is rough (friction is large) and very sharp, which means the radius of curvature of the edge is much smaller than the flat end-face of the rod. Half of the end-face protrudes beyond the table edge, so that when it is released the rod pivots about the edge.

[3] **Problem 12** (Cahn). A tall, thin brick chimney of height L is slightly perturbed from its vertical equilibrium position so that it topples over, rotating rigidly about its base B until it breaks at a point P.



(a) For concreteness, we will model the internal forces in the chimney as shown below. Assume throughout that r is very small.



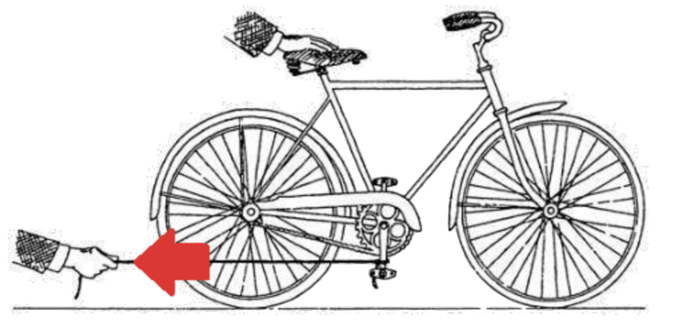
We assume that each piece of the chimney experiences a shear force F and longitudinal tension/compression forces T_1 and T_2 from its neighbors. Find the point on the chimney with the greatest $|T_1|$ or $|T_2|$, assuming the chimney is very thin.

(b) Find the point on the chimney experiencing the greatest shear force F.

(c) At what point is the chimney most likely to break? Do you think the limiting factor is the chimney's maximal compressive strength, tensile strength, or shear strength?

[3] **Problem 13.** IPhO 2014, problem 1A.

[3] **Problem 14** (PPP 14). A bicycle is supported so that it can move forward or backwards but cannot fall sideways; its pedals are in their highest and lowest positions.



A student crouches beside the bicycle and pulls a string attached to the lower pedal, providing a backward horizontal force.

(a) Which way does the bicycle move?

(b) Does the chain-wheel rotate in the same or opposite sense as the rear wheel?

(c) Which way does the lower pedal move relative to the ground?

In particular, be sure to account for the gearing of the bike! To check your answer, watch this video.

[4] **Problem 15.** APhO 2005, problem 1B. A problem on parametric resonance, an idea we first encountered in **M4**. The problem was so subtle that the APhO problem writers themselves could not agree on what the correct answer was!

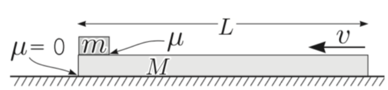
[4] **Problem 16.** INPhO 2020, problem 5. A tough angular collision problem.

[5] **Problem 17.** EuPhO 2019, problem 2. A tough problem about the motion of an rigid body in a magnetic field.

3 Frictional Losses

These miscellaneous problems are grouped under the theme of friction or energy dissipation.

[2] **Problem 18** (Kalda)**.** A plank of length L and mass M lies on a frictionless horizontal surface; on one end sits a small block of mass m.



The coefficient of friction between the block and plank is μ. The plank is sharply hit and given horizontal velocity v. What is the minimum v required for the block to slide across the plank and fall off the other end?

[3] **Problem 19** (BAUPC)**.** A uniform sheet of metal of length ℓ lies on a roof inclined at angle θ, with coefficient of kinetic friction $\mu > \tan\theta$. During the daytime, thermal expansion causes the sheet to uniformly expand by an amount $\Delta\ell \ll \ell$. At night, the sheet contracts back to its original length. What is the displacement of the sheet after one day and night?

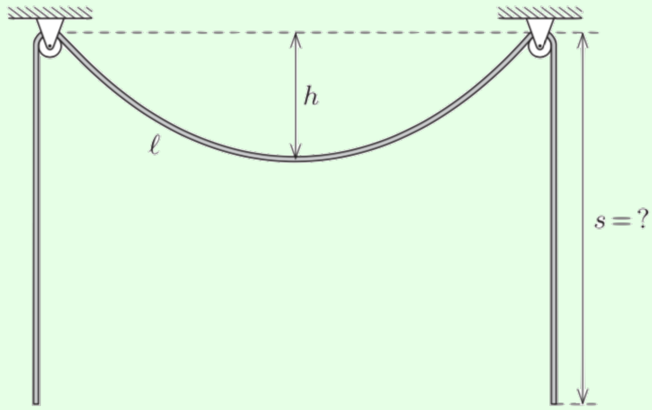
[3] **Problem 20.** APhO 2010, problem 1A. A question about a different kind of inelastic collision.

[5] **Problem 21.** IdPhO 2020, problem 2. A nice problem on anisotropic friction.

4 Ropes, Wires, and Chains

Example 6: MPPP 78

A uniform flexible rope passes over two small frictionless pulleys mounted at the same height.



The length of rope between the pulleys is ℓ, and its sag is h. In equilibrium, what is the length s of the rope segments that hang down on either side?

Solution

The problem can be attacked by differential equations, but there is an elegant solution using only algebra. We let our unknowns be s, the tension $\mathbf{T}_1 = (T_{1,x}, T_{1,y})$ in the rope at the pulley, and the tension T_2 at the lowest point.

Considering the entire sagging portion as the system, vertical force balance gives

$$2T_{1,y} = \lambda \ell g, \quad T_{1,y} = \lambda \ell g/2.$$

Now consider half of the sagging portion as the system. Horizontal force balance gives

$$T_2 = T_{1,x}.$$

Finally, consider one of the hanging portions as the system. Then

$$T_1 = \lambda g s.$$

We hence have three equations, but four unknowns.

For the final equation, we need to consider how the tension changes throughout the rope. This would usually be done by a differential equation, but there is a clever approach using conservation of energy. Suppose we cut the rope somewhere, pull out a segment dx, and reattach the two ends. This requires work $T\,dx$, where T is the magnitude of the local tension. Now suppose we cut the rope somewhere else, separate the ends by dx, and paste our segment inside. This requires work $-T'\,dx$. After this process, the rope is exactly in the same state it was before, so the total work done must be zero.

This would seem to prove that $T = T'$, which is clearly wrong. The extra contribution is that if the two locations have a difference in height Δy, then it takes work $\lambda g(\Delta y)\,dx$ to move the segment from the first to the second. So in equilibrium, for any two points of the rope,

$$\Delta T = \lambda g \, \Delta y.$$

Therefore, we have

$$T_1 - T_2 = \lambda g h.$$

Now we're ready to solve. We have

$$T_1^2 - T_2^2 = (\lambda \ell g/2)^2$$

from our first three equations, and dividing by this new relation gives

$$T_1 + T_2 = \lambda g \, \frac{\ell^2}{4h}.$$

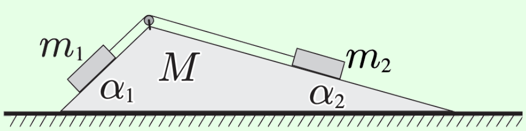
This allows us to solve for T_1, which gives

$$s = \frac{T_1}{\lambda g} = \frac{h}{2} + \frac{\ell^2}{8h}.$$

This is a useful result in real engineering projects: it means that the tension in a cable can be estimated by seeing how much it sags.

Example 7: Kalda 27

A wedge with mass M and acute angles α_1 and α_2 lies on a horizontal surface. A string has been drawn across a pulley situated at the top of the wedge, and its ends are tied to blocks with masses m_1 and m_2.



There is no friction anywhere. What is the acceleration of the wedge?

Solution

This is a classic example of a problem best solved with the Lagrangian-like techniques of **M4**. By working in generalized coordinates, we won't have to solve any systems of equations.

Let s be the distance the rope moves through the pulley, so that both blocks have speed $\dot{s}$ in the noninertial frame of the wedge. The "generalized force" is

$$F_{\text{eff}} = -\frac{dV}{ds} = (m_1 \sin\alpha_1 - m_2 \sin\alpha_2) g.$$

Now, the kinetic energy in the lab frame will be of the form

$$K = \frac{1}{2} M_{\text{eff}} \dot{s}^2$$

which means that, by the Euler–Lagrange equations,

$$\ddot{s} = \frac{F_{\text{eff}}}{M_{\text{eff}}}.$$

Our task is now to calculate M_{eff}. Since the center of mass of the system can't move horizontally, the wedge has speed

$$v_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \dot{s}.$$

Now, it's a bit annoying to directly compute the kinetic energy K in the lab frame, but it's easy to compute the kinetic energy in the frame of the wedge: it's simply $(m_1 + m_2)\dot{s}^2/2$. But the two are also related simply,

$$K + \frac{1}{2}(M + m_1 + m_2) v_w^2 = \frac{1}{2}(m_1 + m_2)\dot{s}^2.$$

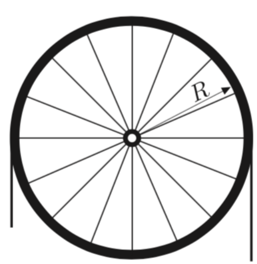
Using this to solve for K, we conclude

$$M_{\text{eff}} = m_1 + m_2 - \frac{(m_1 \cos\alpha_1 + m_2 \cos\alpha_2)^2}{M + m_1 + m_2}.$$

Finally, the desired answer is

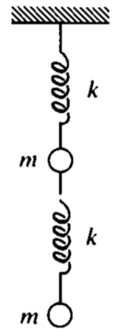
$$a_w = \frac{m_1 \cos\alpha_1 + m_2 \cos\alpha_2}{M + m_1 + m_2} \ddot{s}.$$

[3] **Problem 22** (Kalda). A rope of mass per unit length ρ and length L is thrown over a pulley so that the length of one hanging end is ℓ. The rope and pulley have enough friction so that they do not slip against each other.



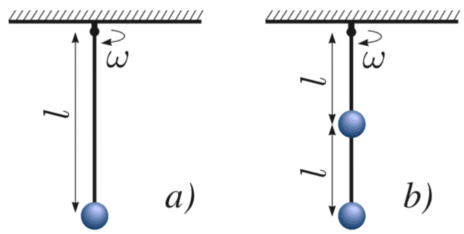
The pulley is a hoop of mass m and radius R attached to a horizontal axle by light spokes. Find the force on the axle immediately after the motion begins.

[3] **Problem 23** (French 5.10). Two equal masses are connected as shown with two identical massless springs of spring constant k.



Considering only motion in the vertical direction, show that the ratio of the frequencies of the two normal modes is $(\sqrt{5}+1)/(\sqrt{5}-1)$.

[3] **Problem 24** (Kalda). A massless rod of length ℓ is attached to the ceiling by a hinge which allows the rod to rotate in a vertical plane.

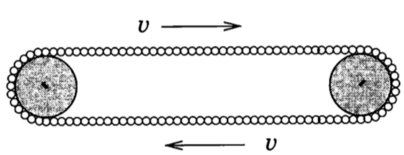


The rod is initially vertical and the hinge is spun with a fixed angular velocity ω.

(a) If a mass m if attached to the bottom of the rod, find the maximum ω for which the configuration is stable.

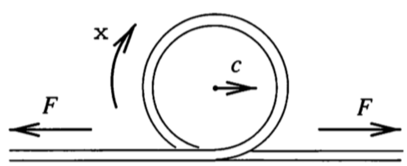
(b) [A] Now suppose another mass m and rod of length ℓ is attached to the first mass by an identical hinge that turns in the same direction, as shown above. Find the maximum ω for which the configuration is stable. (Hint: the configuration is unstable if *any* infinitesimal change in the angles of the rods can lower the energy.)

[3] **Problem 25** (PPP 104). A flexible chain of uniform density is wrapped tightly around two cylinders as shown.



The cylinders are made to rotate and cause the chain to move with speed v. The chain suddenly slips off the cylinders and falls vertically. How does the shape of the chain vary during the fall?

[4] **Problem 26** (PPP 106). A long, heavy flexible rope with mass ρ per unit length is stretched by a constant force F. A sudden movement causes a circular loop to form at one end of the rope.



The center of the loop moves with speed c as shown.

(a) Calculate the speed c, assuming gravity is negligible.

(b) Find the energy E carried by a loop rotating with angular frequency ω.

(c) Show that the momentum p carried by the loop obeys $E = pc$. This is true for waves in general, as we'll see in **W1**.

(d) Find the angular momentum carried by the loop.

5 [A] Advanced Mathematical Techniques

The following problems were cut from earlier problem sets because they required more advanced math; however, they illustrate some very neat and important ideas.

[3] **Problem 27.** In **P1**, you found a general expression for the period of a pendulum oscillating with amplitude θ_0 in terms of an integral, then approximated the integral to find

$$\omega = \omega_0 \left(1 - \frac{\theta_0^2}{16} + O(\theta_0^4)\right)$$

where $\omega_0 = \sqrt{g/L}$. In this problem, we will show a different way to get the same answer, by solving the equation of motion approximately. We write the solution $\theta(t)$ as a series in θ_0. The overall solution is of order θ_0, and the corrections only depend on θ_0^2, so we can write

$$\theta(t) = \theta_0 f_0(t) + \theta_0^3 f_1(t) + \theta_0^5 f_2(t) + \ldots$$

where all the functions $f_i(t)$ are of order 1. Then we plug this expansion into Newton's second law, $\ddot{\theta} + \omega_0^2 \sin\theta = 0$, and expand it out order by order in θ_0.

(a) A naive first guess is to set $f_0(t)$ so that it cancels precisely the order θ_0 terms in this equation, then set $f_1(t)$ to cancel the order θ_0^3 terms, and so on. Using this guess, show that

$$\ddot{f}_0 + \omega_0^2 f_0 = 0, \quad \ddot{f}_1 + \omega_0^2 f_1 = \frac{\omega_0^2 f_0^3}{6}$$

where the first equation has solution $f_0(t) = \cos(\omega_0 t)$.

Unfortunately, this decomposition is not very useful. The problem is that two things are going on at once: the oscillations are not quite sinusoidal, and they have an angular frequency lower than ω_0. The expansion we've done would be useful if we only had the first effect, because then $f_1(t)$ would just capture the small, non-sinusoidal corrections to $f_0(t)$. But our method can't account for the frequency shift; by construction, $f_0(t)$ always oscillates at angular frequency ω_0. Over time, the real oscillation $\theta(t)$ gets out of phase with $f_0(t)$. This manifests itself as a "secular growth" in $f_1(t)$, i.e. it increases in magnitude every cycle until it has a huge value, of order $1/\theta_0^2$, and our perturbative expansion breaks down.

(b) Write the right-hand side of the differential equation for $f_1(t)$ as a sum of sinusoids, and show that it contains a term proportional to $\cos(\omega_0 t)$. This resonantly drives $f_1(t)$, causing the secular growth.

(c) We can salvage our perturbative expansion using the method of "renormalized" frequencies. We impose by fiat that $f_0(t)$ oscillates at the true angular frequency, letting

$$\ddot{f}_0 + \omega^2 f_0 = 0, \quad \omega = \omega_0(1 - c\theta_0^2 + O(\theta_0^4)).$$

Because of this choice, the differential equation for $f_1(t)$, which contains all terms at order θ_0^3, will be altered. The correct choice of ω is precisely the one for which this eliminates the secular growth of $f_1(t)$. Using this idea, show that $c = 1/16$.

If you keep going, you'll find the next term $f_2(t)$ still has secular growth. We can remove it by having both $f_0(t)$ and $f_1(t)$ oscillate at frequency $\omega_0(1 - \theta_0^2/16 + d\theta_0^4)$, where d is chosen to cancel the secular growth of $f_2(t)$. In this way, the frequency can be found to any order in θ_0^2. (This technique is called the method of strained coordinates. It's an example of multiple-scale analysis.)

[3] **Problem 28.** You might be wondering how we can solve the weakening spring problem from **M4** without anything fancy like the adiabatic theorem. There is a general technique to solve linear differential equations whose coefficients are slowly varying. First, write the equation of motion as

$$\ddot{x} + \omega^2(t)x = 0.$$

Then expand $x(t)$ as

$$x(t) = A(t)e^{i\phi(t)}, \quad \dot{\phi}(t) = \omega(t).$$

The point of writing $x(t)$ this way is that pulling out the factor of $e^{i\phi(t)}$ will automatically account for the rapid oscillations. The factor $A(t)$ only varies slowly, so it's easier to handle by itself.

(a) Evaluate $\ddot{x}(t)$ and plug it into the equation of motion.

(b) Using the fact that $A(t)$ and $\omega(t)$ vary slowly, throw out small terms in your equation from part (a), until you get a differential equation you can easily integrate. This is a simple example of the WKB approximation.

(c) Show that this gives the expected final result for a weakening spring.

[4] **Problem 29** (BAUPC 1996)**.** A mass M is located at the vertex of an angle $\theta \ll 1$ formed by two massless sticks of length ℓ. The structure is held so that the left stick is initially vertical, then released. The right stick hits the ground at time $t = 0$. The structure then rocks back and forth, coming to a stop at time $t = T$.

(a) Prove the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$$

using the result $\sum_{n \geq 1} 1/n^2 = \pi^2/6$.

(b) Calculate T to leading order in θ. Performing the expansion is rather tricky and requires the previous identity.

[3] **Problem 30.** In this problem, we'll go through Laplace's slick derivation of Kepler's first law. Throughout, we assume the orbit takes place in the xy plane, with the Sun at the origin.

(a) Show that

$$\ddot{x} = -\frac{\gamma x}{r^3}, \quad \ddot{y} = -\frac{\gamma y}{r^3}$$

where γ is a constant that depends on the parameters.

(b) Show that

$$\frac{d}{dt}(r^3 \ddot{x}) = -\gamma \dot{x}, \quad \frac{d}{dt}(r^3 \ddot{y}) = -\gamma \dot{y}.$$

(c) Show that

$$\frac{d}{dt}(r^3 \ddot{r}) = -\gamma \dot{r}.$$

(Hint: this can get messy. As a first step, try showing the left-hand side is equal to $(r^2/2)\, d^3(r^2)/dt^3$. You will have to switch variables to x and y and then switch back; for these purposes it's useful to use the results of part (a), and the definition $r^2 = x^2 + y^2$.)

(d) Define $\psi(t) = r(t)^3$. In parts (b) and (c), we have shown that the differential equation

$$\frac{d}{dt}\left(\psi(t)\,\frac{du}{dt}\right) = -\gamma u$$

has three solutions, namely $\dot{x}$, $\dot{y}$, and $\dot{r}$. Any second-order linear differential equations only has two independent solutions. If $\dot{x}$ and $\dot{y}$ are not independent, the orbit is simply a line, which is trivial. Assuming that doesn't happen, they are independent, so $\dot{r}$ must be a linear combination of them,

$$\dot{r} = A\dot{x} + B\dot{y}.$$

Use this result to argue that the orbit is a conic section.

6 Mechanics and Geometry

For dessert, we'll consider a few cute problems that relate statics to geometry.

Example 8

Given a triangle ABC, the Fermat point is the point X that minimizes $AX + BX + CX$. Design a machine that finds the Fermat point.

Solution

We take a horizontal plane and drill holes at points A, B, and C. A mass M on a rope is fed through each hole, and the three ends of the rope are tied together at point X. The gravitational potential energy is proportional to $AX + BX + CX$, so in equilibrium X lies on the Fermat point. Moreover, since the tensions in each rope are all equal to Mg, force balance requires $\angle AXB = \angle BXC = \angle CXA = 120°$.

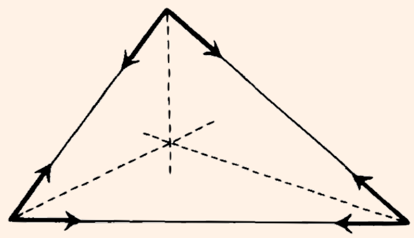
[1] **Problem 31.** Using similar reasoning, design a machine that finds the point X that minimizes $(AX)^2 + (BX)^2 + (CX)^2$. What geometrical property can you conclude about this point?

Example 9

Show that the incenter of a triangle (i.e. the meeting point of the angle bisectors) exists.

Solution

Apply six forces at the vertices of a triangle as shown.



These forces clearly balance, and also produce no net torque on the triangle. Now combine the forces applied at each vertex, yielding three forces that point along the angle bisectors. By the principles of **M2**, the torques of these forces can only balance if their lines of action meet at a point. Therefore the angle bisectors are concurrent, so the incenter exists.

Example 10

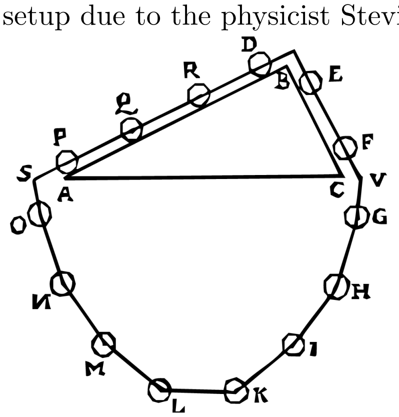
Let AB be a diameter of a circle, and let a mass be free to slide on the circle. The mass is connected to two identical straight springs of zero rest length, which are in turn connected to points A and B. At what points C can the mass be in static equilibrium?

Solution

The potential energy of the system is proportional to $(\mathrm{AC})^2 + (\mathrm{BC})^2$. Since ABC is a right triangle, this is just equal to $(\mathrm{AB})^2$ by the Pythagorean theorem. Since the potential energy doesn't depend on where the mass is, it can be at static equilibrium at any point on the circle. Alternatively, you can show that the mass is in static equilibrium by force balance, and use the reasoning in reverse to derive the Pythagorean theorem.

[1] **Problem 32.** Consider a right triangle ABC filled with a fluid of uniform pressure. Using torque balance, establish the Pythagorean theorem.

[1] **Problem 33.** Shown below is a setup due to the physicist Stevin.



One might argue that because there are more masses on AB than on BC, this is a perpetual motion machine that turns counterclockwise. By using the fact that perpetual motion machines don't actually exist, prove the law of sines.

[2] **Problem 34.** Consider the n-sided polygon P of least possible area that circumscribes a closed convex curve K. Prove that every tangency point of K with a side of P is the midpoint of that side. (Hint: begin by supposing that the area outside P is filled with a gas of uniform pressure, with a vacuum inside P.)

[2] **Problem 35.** In this problem we'll derive Kepler's first law yet again, using no calculus, but a bit of Euclidean geometry. As usual, we suppose a planet of mass m orbits a fixed star of much greater mass M. Placing the star at the origin, let ϕ be the angle between $\mathbf{r}$ and $\mathbf{v}$ for the planet.

(a) Write down the quantities E and L in terms of G, M, m, v, r, and ϕ, and show that

$$\left(r^2 + \frac{GMm}{E} r\right) \sin^2 \phi = \frac{L^2}{2mE}.$$

(b) Now consider an ellipse with semimajor axis a and eccentricity e, meaning that the distance between the foci is $2ae$, with one of the foci F at the origin. Consider a point P on the ellipse, so that the angle between the tangent to the ellipse at P and FP is ϕ. If $r = |FP|$, show that

$$(r^2 - 2ar) \sin^2 \phi = -a^2(1 - e^2).$$

You will have to use the geometrical property that a light ray sent from one focus will reflect at the ellipse to hit the other focus.

(c) By comparing your results for (a) and (b), conclude that the orbit is an ellipse with

$$a = -\frac{GMm}{2E}, \quad e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}.$$