PHYS-330: QFT I — Solutions 8

Stanford University, 2022 — Bernhard Mistlberger, Kevin Zhou

1. Magnetic moments in quantum electrodynamics. (15 points)

The interaction Hamiltonian in quantum electrodynamics for a Dirac field of charge q is

$$H_I = q \int d^3x\, \bar{\Psi}\gamma^\mu \Psi A_\mu. \tag{1}$$

To understand the physical meaning of this expression, we can evaluate its matrix elements in states $|p,s\rangle = \sqrt{2E_p}\, a_s^\dagger(p)|0\rangle$ with a single fermion. We assume the electromagnetic field is in a quantum state with negligible field uncertainty, so that its field operator A_μ can be replaced with a time-independent classical expectation value $A_\mu^{\rm cl}(\mathbf{x})$.

a) Show that the matrix elements of the Schrodinger picture Hamiltonian are

$$\langle p,s|H_I|p',s'\rangle = q\int d^3x\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\gamma^\mu u_{s'}(p')A_\mu^{\rm cl}(\mathbf{x}) \tag{2}$$

when $|p',s'\rangle \neq |p,s\rangle$.

Solution: Plugging in the Schrodinger picture mode expansion of the Dirac field, and using the abbreviated notation introduced in the solutions to problem set 2,

$$\langle p,s|H_I|p',s'\rangle = q\sum_{r,r'}\int d\mathbf{x}\frac{đ\mathbf{q}}{\sqrt{2E_q}}\frac{đ\mathbf{q}'}{\sqrt{2E_{q'}}}\sqrt{2E_p 2E_{p'}}$$
$$\times\langle 0|a_s(\mathbf{p})\left(a_r^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{u}_r(q)+b_r(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\bar{v}_r(q)\right)\not{A}\left(a_{r'}(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{x}}u_{r'}(q)+b_{r'}^\dagger(\mathbf{q}')e^{-i\mathbf{q}'\cdot\mathbf{x}}v_{r'}(q')\right)a_{s'}^\dagger(\mathbf{p}')|0\rangle. \tag{S1}$$

Now let's think about the structure of this matrix element. We can only get nonzero contributions from the terms $a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')$ and $b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')$. On the other hand, the latter term yields

$$\langle 0|a_s(\mathbf{p})b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle = \delta_{ss'}\delta_{rr'}đ(\mathbf{p}-\mathbf{p}')đ(\mathbf{q}-\mathbf{q}') \tag{S2}$$

and therefore cannot contribute matrix elements with $|p',s'\rangle \neq |p,s\rangle$. Discarding this term, we're left with

$$\langle p,s|H_I|p',s'\rangle = q\int d\mathbf{x}đ\mathbf{q}đ\mathbf{q}'\frac{\sqrt{2E_p 2E_{p'}}}{\sqrt{2E_q 2E_{q'}}}e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r,r'}\bar{u}_r(q)\not{A}u_{r'}(q')\langle 0|a_s(\mathbf{p})a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle \tag{S3}$$

$$= q\int d\mathbf{x}đ\mathbf{q}'\frac{\sqrt{2E_{p'}}}{\sqrt{2E_{q'}}}e^{-i(\mathbf{p}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r'}\bar{u}_s(p)\not{A}u_{r'}(q')\langle 0|a_{r'}(\mathbf{q}')a_{s'}^\dagger(\mathbf{p}')|0\rangle \tag{S4}$$

$$= q\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\not{A}u_{s'}(p'). \tag{S5}$$

This is the desired result. Using the Gordon identity we find

$$\langle p,s|H_I|p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_\mu + p_\mu + i\sigma_{\mu\nu}(p'^\nu - p^\nu)\right)u_{s'}(p')A^\mu(x) \tag{S6}$$

In nonrelativistic quantum mechanics, the single-particle states are $|\mathbf{p},s\rangle$. If the particle has charge q' and g-factor g, then its magnetic moment is $\mu = (gq'/2m)\,\mathbf{S}$ where $\mathbf{S}$ is the spin. In terms of the particle's position $\mathbf{x}$, the Hamiltonian contains the terms

$$H_I^{\rm nr} = q'A^0(\mathbf{x}) - \boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{x}). \tag{3}$$

To find the values of q' and g, we equate matrix elements in the nonrelativistic limit,

$$\langle p, s|H_I|p', s'\rangle = 2m\langle \mathbf{p}, s|H_I^{\rm nr}|\mathbf{p}', s'\rangle \text{ when } |\mathbf{p}|, |\mathbf{p}'| \ll m \tag{4}$$

where the factor of $2m$ converts between relativistic and nonrelativistic normalization. We could evaluate both sides for general $A_\mu^{\rm cl}(\mathbf{x})$, but it is easier to consider two special cases. In both cases it will be helpful to use the Gordon identity from set 7.

b) Evaluate both sides of (4) in a static electric field, corresponding to general $A^0(\mathbf{x})$ and $\mathbf{A} = 0$. Show that they agree when $q' = q$, as one would expect.

Solution: If we set $\mathbf{A} = 0$, then the matrix element above reduces to

$$\langle p, s|H_I|p', s'\rangle = \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p_0' + p_0 + i\sigma_{0i}(p'^i - p^i)\right)u_{s'}(p')A^0(x) \tag{S7}$$

since σ_{00} vanishes. In the nonrelativistic limit, $p_0, p_0' \approx m$ are much greater than p^i, p'^i, so we may simply drop the latter terms, giving

$$\langle p, s|H_I|p', s'\rangle = q\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)u_{s'}(p')A^0(x) \tag{S8}$$

$$= 2m\, q\delta_{ss'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}A^0(x). \tag{S9}$$

On the other hand, in nonrelativistic quantum mechanics we have

$$2m\langle \mathbf{p}, s|q'A^0(\hat{\mathbf{x}})|\mathbf{p}', s'\rangle = 2m\, q'\delta_{ss'}\langle \mathbf{p}|A^0(\hat{\mathbf{x}})|\mathbf{p}'\rangle \tag{S10}$$

$$= 2m\, q'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}\,\langle \mathbf{p}|\mathbf{y}\rangle\langle \mathbf{y}|A^0(\hat{\mathbf{x}})|\mathbf{z}\rangle\langle \mathbf{z}|\mathbf{p}'\rangle \tag{S11}$$

$$= 2m\, q'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}\, e^{-i\mathbf{p}\cdot\mathbf{y}}e^{i\mathbf{p}'\cdot\mathbf{z}}\delta(\mathbf{y}-\mathbf{z})A^0(\mathbf{z}) \tag{S12}$$

$$= 2m\, q'\delta_{ss'}\int d\mathbf{y}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{y}}A^0(\mathbf{y}) \tag{S13}$$

$$= 2m\, q'\delta_{ss'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}A^0(\mathbf{x}). \tag{S14}$$

The two matrix elements match if $q = q'$, as desired.

c) Show that if the fermion was a classical spinning ball with uniform mass and charge density, then its g-factor would be 1. This strongly disagrees with the measured value.

Solution: For simplicity, let's consider a thin ring of radius r, angular velocity ω, mass m, and charge q. Then the angular momentum is

$$S = mvr = m\omega r^2 \tag{S15}$$

and the magnetic moment is

$$\mu = IA = \frac{q}{2\pi/\omega}(\pi r^2) = \frac{q\omega r^2}{2} \tag{S16}$$

Therefore, the ring has $\mu/S = q/2m$, corresponding to a g-factor of 1. Since a ball is just a superposition of such rings, this shows that $g = 1$ for a ball, or more generally any spinning object with axial symmetry.

d) Evaluate both sides of (4) in a static magnetic field, corresponding to $A^0 = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}(\mathbf{x})$, and infer the value of g. (Hint: to relate spin in the Dirac theory to spin in the nonrelativistic theory, recall that in the nonrelativistic theory, spinors ξ_s have two components and the spin operator is $\mathbf{S} = \boldsymbol{\sigma}/2$. You already showed in problem set 7 how the Dirac spinors $u_s(p)$ are built from ξ_s. You will have to integrate by parts, so assume $\mathbf{A}$ and $\mathbf{B}$ vanish at infinity.)

Solution: In this case, since A^0 vanishes, we have

$$\langle p, s|H_I|p', s'\rangle = \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_i + p_i + i\sigma_{i\nu}(p'^\nu - p^\nu)\right)u_{s'}(p')A^i(x). \tag{S17}$$

This might look more complicated than what we had before using the Gordon identity, but this form is useful because it lets us isolate the physically distinct effects of different terms, and figure out which to keep in the nonrelativistic limit. (Though of course you could also do the problem without using the Gordon identity too.) Now, we need to choose which term to keep within the parentheses.

- The p_i and p'_i terms are order mv.
- The term $\sigma_{i0}(p'^0 - p^0)$ is not order m, since the m's cancel. Instead, it's order mv^2.
- The term $\sigma_{ij}(p'^j - p^j)$ is order mv.

The second contribution is therefore negligible in the nonrelativistic limit. The first contribution is not negligible, but it also has nothing to do with spin: it will just yield the $\mathbf{p}\cdot\mathbf{A}$ term in the nonrelativistic Hamiltonian that physically corresponds to the magnetic force on a charged particle. (Considering it won't tell us anything new, because we already know that $q' = q$ from part (b).) We therefore focus on the third term, giving

$$\langle p, s|H_I|p', s'\rangle \supset \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(i\sigma_{ij}(p'^j - p^j)\right)u_{s'}(p')A^i(x) \tag{S18}$$

$$= \frac{q}{2m}\int d\mathbf{x}\; e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}(p^j - p'^j)(\bar{u}_s(p)\frac{[\gamma^i,\gamma^j]}{2}u_{s'}(p'))A^i(x). \tag{S19}$$

To make further progress, we use the results we derived in the Weyl representation in problem set 7. First, because we're only considering the leading term in the nonrelativistic limit, it suffices to expand the spinor solutions to lowest order in m,

$$u_s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} \tag{S20}$$

We can also explicitly evaluate the commutators. For example, we have

$$[\gamma^1,\gamma^2] = -2i\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix} \tag{S21}$$

from which we conclude in general that

$$[\gamma^i,\gamma^j] = -2i\epsilon^{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}. \tag{S22}$$

Plugging these results in and simplifying, we find

$$\langle p, s|H_I|p', s'\rangle \supset q\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}(-i)(p^j - p'^j)\left(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'}\right)A^i(x). \tag{S23}$$

To handle the $p^j - p'^j$, we write it as a derivative and integrate by parts, dropping a boundary term,

$$\langle p, s|H_I|p', s'\rangle \supset q\int d\mathbf{x}\,\frac{\partial}{\partial x_j}\left(e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\right)\left(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'}\right)A^i(x) \tag{S24}$$

$$= q\,\xi_s^\dagger\sigma^k\xi_{s'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\epsilon^{ijk}\partial^j A^i(x) \tag{S25}$$

$$= -q\,\xi_s^\dagger\sigma^k\xi_{s'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}B^k. \tag{S26}$$

We can now compare this to the nonrelativistic result,

$$2m\langle\mathbf{p}, s|(-\boldsymbol{\mu}\cdot\mathbf{B})|\mathbf{p}', s'\rangle = -gq'\,\xi_s^\dagger S^k\xi_{s'}\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}B^k \tag{S27}$$

by the same logic as in part (b). Since $q' = q$ and $\mathbf{S} = \boldsymbol{\sigma}/2$, we conclude that $g = 2$.

The agreement of this value you found in part (d) with the experimentally measured value for the electron was one of the early triumphs of the Dirac equation.

2. Decays of the Higgs boson. (10 points)

The Standard Model contains three charged leptons, the electron e, muon μ, and tau τ, which are described by Dirac fields and differ only by their mass. It also contains a spinless particle called the Higgs boson, described by a real scalar field h. The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2}m_h^2\, h^2 + \sum_i \bar{\Psi}_i(i\partial\!\!\!/ - m_i)\Psi_i \tag{5}$$

where $i \in \{e, \mu, \tau\}$, and the numeric values of the masses are

$$m_h = 125\,\text{GeV}, \qquad m_e = 511\,\text{keV}, \qquad m_\mu = 105.7\,\text{MeV}, \qquad m_\tau = 1777\,\text{MeV}. \tag{6}$$

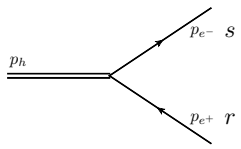
The Higgs field couples to the charged leptons by a Yukawa coupling proportional to mass,

$$\mathcal{L}_{\text{int}} = -\sum_i \frac{m_i}{v}\, h\bar{\Psi}_i\Psi_i \tag{7}$$

Here, v is a constant associated with the breaking of electroweak symmetry, but for this problem you will only need its value, $v = 246\,\text{GeV}$. For this problem, all of your final answers should be numeric, and given to at least two significant figures.

a) Compute the partial decay rate for a Higgs boson to an electron-positron pair $\Gamma_{H\to e^+e^-}$ at leading non-vanishing order in perturbation theory, giving your answer in eV. (Hint: sum over final spin states, and reuse results from problem set 5.)

Solution: There is one Feynman diagram, giving matrix element

$\mathcal{M}_{h\to e^+e^-} =$ 

$= \bar{u}_s(p_{e^-})\left(-i\frac{m_e}{v}\right)v_r(p_{e^+}).$ (S28)

Squaring the matrix element and summing over final spin states gives

$$|\bar{\mathcal{M}}_{h\to e^+e^-}|^2 = \frac{m_e^2}{v^2}\sum_{r,s}\bar{u}_s(p_{e^-})v_r(p_{e^+})\bar{v}_r(p_{e^+})u_s(p_{e^-}) \tag{S29}$$

$$= \frac{m_e^2}{v^2}\,\text{tr}\left[(p\!\!\!/_{e^-} + m_e)(p\!\!\!/_{e^+} - m_e)\right] \tag{S30}$$

$$= \frac{m_e^2}{v^2}\left(\text{tr}\left[p\!\!\!/_{e^-}p\!\!\!/_{e^+}\right] - 4m_e^2\right) \tag{S31}$$

$$= 2\frac{m_e^2}{v^2}\left(2p_{e^-}\cdot p_{e^+} - 2m_e^2\right) \tag{S32}$$

$$= 2\frac{m_e^2}{v^2}\left(m_h^2 - 4m_e^2\right). \tag{S33}$$

In the last line we used the fact that

$$m_h^2 = p_h^2 = (p_{e^-} + p_{e^+})^2 = 2m_e^2 + 2p_{e^+}\cdot p_{e^-}. \tag{S34}$$

The decay width of a Higgs to leptons l^+l^- is defined by

$$\Gamma_{h\to l^+l^-} = \frac{1}{2m_h}\int \frac{d^4p_{l^-}}{(2\pi)^4}(2\pi)\delta_+(p_{l^-}^2 - m_l^2)\frac{d^4p_{l^+}}{(2\pi)^4}(2\pi)\delta_+(p_{l^+}^2 - m_l^2) \tag{S35}$$

$$\times\,(2\pi)^4\delta^{(4)}(p_h - p_{l^-} - p_{l^+})|\bar{\mathcal{M}}_{h\to e^+e^-}|^2. \tag{S36}$$

By reusing results from problem set 5, we immediately find

$$\Gamma_{h\to l^+l^-} = \frac{1}{16\pi m_h^2}\sqrt{m_h^2 - 4m_e^2}|\bar{\mathcal{M}}_{h\to e^+e^-}|^2 = \frac{m_e^2 m_h}{8\pi v^2}\left(1 - 4m_e^2/m_h^2\right)^{3/2} = 0.021\text{eV}. \tag{S37}$$

b) Find the ratios $\Gamma_{H\to\mu^+\mu^-}/\Gamma_{H\to e^+e^-}$ and $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to e^+e^-}$.

Solution: Note that since m_e, m_μ, and m_τ are all much less than m_h, we have

$$\Gamma_{h\to l^+l^-} \approx \frac{m_l^2 m_h}{8\pi v^2} \tag{S38}$$

for each charged lepton. That is, the decay rate is proportional to the mass squared, so the Higgs boson is much more likely to decay to heavier leptons,

$$\frac{\Gamma_{H\to\mu^+\mu^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\mu^2}{m_e^2} = 4.3\times 10^4, \quad \frac{\Gamma_{H\to\tau^+\tau^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\tau^2}{m_e^2} = 1.2\times 10^7. \tag{S39}$$

c) Find the probability a Higgs boson decays to $\tau^+\tau^-$, also known as the branching ratio

$$\text{BR}_{H\to\tau^+\tau^-} = \frac{\Gamma_{H\to\tau^+\tau^-}}{\sum_X \Gamma_{H\to X}}. \tag{8}$$

The denominator is the total decay rate of the Higgs boson, equal to 4.1 MeV. (To check your answer, you can consult the so-called Yellow Report.)

Solution: Plugging in the numbers gives $\text{BR}_{H\to\tau^+\tau^-} = 6.3\%$. (The Yellow Report gives a slightly different value because it accounts for higher-order corrections.)

The decay of Higgs bosons to taus was confirmed experimentally only recently [1,2], and first evidence for the Higgs coupling to muons has been detected [3,4]. Establishing the coupling of the Higgs to electrons remains a monumental experimental challenge.

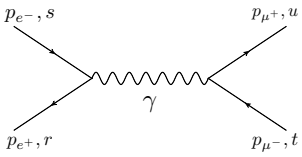
3. Electron-positron annihilation to muons. (15 points)

One of the key successes of quantum electrodynamics is its description of particle creation and annihilation processes at relativistic energies. In this exercise we will consider the process $e^+e^- \to \mu^+\mu^-$, where the electron and muon are Dirac fields of charge e, and mass m_e and m_μ respectively.

a) Let the initial momenta be $p^\mu_{e^+}$ and $p^\mu_{e^-}$ and the final momenta be $p^\mu_{\mu^+}$ and $p^\mu_{\mu^-}$. Find the scattering matrix element $\mathcal{M}$ for this process, to leading nonvanishing order in e.

Solution: There is one relevant Feynman diagram, which gives

$\mathcal{M}_{e^+e^-\to\mu^+\mu^-} =$



(S40)

(S41)

(S42)

$$= \bar{u}_t(p_\mu^-)(-ie\gamma^\mu_{tu})v_u(p_{\mu^+})\left(\frac{-i\eta_{\mu\nu}}{(p_{e^+}+p_{e^-})^2}\right)\bar{v}_r(p_{e^+})(-ie\gamma^\nu_{rs})u_s(p_{e^-}). \tag{S43}$$

b) Let $|\bar{\mathcal{M}}|^2$ be the square of the matrix element, summed over final spin states and averaged over initial spin states. Compute $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, and the Mandelstam variables s and t, where

$$s = (p_{e^+} + p_{e^-})^2, \quad t = (p_{e^-} - p_{\mu^-})^2, \quad u = (p_{e^-} - p_{\mu^+})^2. \tag{9}$$

Solution: Before starting, we note that the Mandelstam variables are simply related to inner products,

$$\frac{s}{2} = m_e^2 + p_{e^+} \cdot p_{e^-} = m_{\mu^2} + p_{\mu^+} \cdot p_{\mu^-} \tag{S44}$$

and

$$t = m_e^2 + m_\mu^2 - p_{e^-} \cdot p_{\mu^-} = m_e^2 + m_\mu^2 - p_{e^+} \cdot p_{\mu^+} \tag{S45}$$

$$u = m_e^2 + m_\mu^2 - p_{e^-} \cdot p_{\mu^+} = m_e^2 + m_\mu^2 - p_{e^+} \cdot p_{\mu^-} \tag{S46}$$

Now, squaring, summing over final spins and averaging over initial spins yields

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{e^4}{s^2}\frac{1}{4}\sum_{\text{spins}} \bar{u}_t(p_\mu^-)\gamma^\mu_{tu} v_u(p_{\mu^+})\bar{v}_{u'}(p_\mu^+)\gamma^\nu_{u't'} u_{t'}(p_{\mu^-}) \\ &\qquad \times \bar{v}_r(p_{e^+})(\gamma_\mu)_{rs} u_s(p_{e^-})\bar{u}_{s'}(p_{e^-})(\gamma_\nu)_{s'r'} v_{r'}(p_{e^+}) \qquad \text{(S47)} \\ &= \frac{e^4}{s^2}\frac{1}{4}\operatorname{tr}\left[(\not{p}_{\mu^-} + m_\mu)\gamma^\mu(\not{p}_{\mu^+} - m_\mu)\gamma^\nu\right]\operatorname{tr}\left[(\not{p}_{e^-} + m_e)\gamma_\mu(\not{p}_{e^+} - m_e)\gamma_\nu\right]. \qquad \text{(S48)} \end{aligned}$$

Both traces have the same form, and can be evaluated using the results from problem set 6,

$$\begin{aligned} \operatorname{tr}\left[(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma^\nu\right] &= \operatorname{tr}[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu] - m^2\operatorname{tr}[\gamma^\mu\gamma^\nu] \qquad \text{(S49)} \\ &= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)\eta^{\mu\nu}) - 4m^2\eta^{\mu\nu} \qquad \text{(S50)} \\ &= 4(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{s}{2}\eta^{\mu\nu}) \qquad \text{(S51)} \end{aligned}$$

where in the last step we used (S44). Inserting this result twice above, we have

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{4e^4}{s^2}\left(p^\mu_{e^+}p^\nu_{e^-} + p^\mu_{e^-}p^\nu_{e^+} - \frac{s}{2}\eta^{\mu\nu}\right)\left(p_{\mu^+,\mu}p_{\mu^-,\nu} + p_{\mu^-,\mu}p_{\mu^+,\nu} - \frac{s}{2}\eta_{\mu\nu}\right) \qquad \text{(S52)} \\ &= \frac{4e^4}{s^2}\left(s^2 - s(p_{\mu^+}\cdot p_{\mu^-} + p_{e^+}\cdot p_{e^-}) + 2(p_{e^+}\cdot p_{\mu^+})(p_{e^-}\cdot p_{\mu^-}) + 2(p_{e^+}\cdot p_{\mu^-})(p_{e^-}\cdot p_{\mu^+})\right). \qquad \text{(S53)} \end{aligned}$$

We can then write the inner products in terms of s, t, and u using (S44) and (S45). There are many possible forms for the answer, since

$$s + t + u = 2m_e^2 + 2m_\mu^2. \tag{S54}$$

If we eliminate u, then we get

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[(s+t)^2 + t^2 - 4t(m_e^2 + m_\mu^2) + 2(m_e^2 + m_\mu^2)^2\right]. \tag{S55}$$

Alternatively, if we eliminate s inside the brackets, then we get

$$|\bar{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[(u - 2(m_e^2 + m_\mu^2))^2 + (t - 2(m_e^2 + m_\mu^2))^2 - 2(m_e^2 + m_\mu^2)^2\right]. \tag{S56}$$

For the rest of this problem, we specialize to the center of mass frame.

c) Rewrite $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, s, and the angle θ between $\mathbf{p}_{e^-}$ and $\mathbf{p}_{\mu^-}$.

Solution: In the centre-of-mass frame we have

$$E_{e^-} = E_{e^+}, \quad E_{\mu^-} = E_{\mu^+}, \quad \mathbf{p}_{e^-} = -\mathbf{p}_{e^+}, \quad \mathbf{p}_{\mu^-} = -\mathbf{p}_{\mu^+} \tag{S57}$$

which implies that

$$s = 4E_{e^-}^2 = 4E_{\mu^-}^2. \tag{S58}$$

Furthermore, we have $E_{e^\pm}^2 = |\mathbf{p}_{e^\pm}|^2 + m_e^2$ and $E_{\mu^\pm}^2 = |\mathbf{p}_{\mu^\pm}|^2 + m_\mu^2$.

Starting from (S55), we need to eliminate t in favor of θ. Starting from the definition of t,

$$t = m_e^2 + m_\mu^2 - 2E_{e^-}E_{\mu^-} + 2|\mathbf{p}_{e^-}||\mathbf{p}_{\mu^-}|\cos\theta \tag{S59}$$

$$= \frac{1}{2}\left(2m_e^2 + 2m_\mu^2 - s + \sqrt{s-4m_e^2}\sqrt{s-4m_\mu^2}\cos\theta\right). \tag{S60}$$

Plugging this into (S55) and simplifying gives

$$|\bar{\mathcal{M}}|^2 = e^4\left[1 + \frac{4(m_e^2+m_\mu^2)}{s} + (1-4m_e^2/s)(1-4m_\mu^2/s)\cos^2\theta\right] \tag{S61}$$

Also note that solving (S60) for $\cos\theta$ gives

$$\cos\theta = 2\,\frac{t - m_e^2 - m_\mu^2 + \frac{s}{2}}{\sqrt{s-4m_e^2}\sqrt{s-4m_\mu^2}}. \tag{S62}$$

d) Starting from equation (4.84) of Peskin and Schroeder, compute the differential cross section $d\sigma/d\Omega$ and the total cross section.

Solution: Equation (4.84) of Peskin and Schroeder is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{e^-}\,2E_{e^+}\,|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}|}\frac{|\mathbf{p}_{\mu^-}|}{(2\pi)^2\,4E_{\rm cm}}|\bar{\mathcal{M}}|^2. \tag{S63}$$

Our first task is to write the phase space factors in the same variables we've been using above. We note that $\sqrt{s} = E_{\rm cm} = 2E_{e^-} = 2E_{e^+}$ in the center of mass frame. Furthermore,

$$|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}| = 2|\mathbf{v}_{e^-}| = 2\frac{|\mathbf{p}_{e^-}|}{E_{e^-}} = 2\frac{\sqrt{E_{e^-}^2 - m_{e^-}^2}}{E_{e^-}} = 2\sqrt{1-4m_e^2/s} \tag{S64}$$

and similarly

$$|\mathbf{p}_{\mu^-}| = \frac{\sqrt{s}}{2}\sqrt{1-4m_\mu^2/s}. \tag{S65}$$

Plugging these results in and simplifying gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}|\bar{\mathcal{M}}|^2. \tag{S66}$$

To find the total cross section, we integrate over solid angle, noting that

$$\int d\Omega = 4\pi, \quad \int \cos^2\theta\, d\Omega = \int_0^{2\pi} d\phi \int_0^\pi d\theta\, \sin\theta\cos^2\theta = \frac{4\pi}{3}. \tag{S67}$$

The final result is

$$\sigma = \frac{e^4}{16\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}\left[1 + \frac{4(m_e^2+m_\mu^2)}{s} + \frac{(1-4m_e^2/s)(1-4m_\mu^2/s)}{3}\right] \tag{S68}$$

$$= \frac{e^4}{12\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}(1+2m_e^2/s)(1+2m_\mu^2/s). \tag{S69}$$

As a check, we recover the expected result in the ultrarelativistic limit,

$$\sigma \approx \frac{e^4}{12\pi s} = \frac{4\pi\alpha^2}{3s} \tag{S70}$$

which matches the result in Peskin and Schroeder.

You can see section 5.1 of Peskin and Schroeder to get started or check your answer. But note that the book neglects the mass of the electron, while here we account for it.

4. ⋆ Nonminimal couplings. (5 points)

In problem 1, you found the charge and g-factor for a Dirac fermion minimally coupled to the electromagnetic field, i.e. via the simplest possible interaction (1). However, if the fermion is composite, or interacts with heavier particles, we might need additional terms to describe the coupling. For all parts of this problem, you should adopt the formalism of problem 1 and work in the nonrelativistic limit. Detailed calculations are not needed; qualitative final answers (with justification) are sufficient.

a) The g-factors of the proton and neutron are *not* given by the result you found in problem 1. When the Dirac equation was invented, physicists explained this by adding an additional term to $\mathcal{H}_I$, proportional to $i\bar{\Psi}[\gamma^\mu, \gamma^\nu]\Psi\, F_{\mu\nu}$. Show that the physical effect of such a term is indeed to shift the magnetic dipole moment.

Solution: If we follow the exact same logic as in problem 1, we'll end up with a term in $\langle p, s|H_I|p', s'\rangle$ proportional to

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)[\gamma^\mu, \gamma^\nu]u_{s'}(p')F_{\mu\nu}(x). \tag{S71}$$

In the presence of a uniform magnetic field, this becomes

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)[\gamma^i, \gamma^j]u_{s'}(p')F_{ij}(x) \propto \int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'}F_{ij} \tag{S72}$$

$$\propto \delta(\mathbf{p}-\mathbf{p}')\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{B})\xi_{s'} \tag{S73}$$

which, as we already saw, is precisely what you would get from a magnetic dipole moment.

You might be concerned that this term also causes an interaction with electric fields, as it contains

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)[\gamma^0, \gamma^i]u_{s'}(p')F_{0i}(x). \tag{S74}$$

However, we have

$$[\gamma^0, \gamma^i] = -2\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix} \tag{S75}$$

so that when we do the spinor contractions in the nonrelativistic limit, we get two terms that cancel in the nonrelativistic limit. So the only effect of this term is a magnetic dipole moment.

b) What is the physical effect of a term proportional to $\bar{\Psi}[\gamma^\mu, \gamma^\nu]\gamma^5\Psi\, F_{\mu\nu}$?

Solution: By the same logic as before, we'll get a term in the matrix element like

$$\int d\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)[\gamma^\mu, \gamma^\nu]\gamma^5 u_{s'}(p')F_{\mu\nu}(x) \tag{S76}$$

where

$$\gamma^5 = \begin{pmatrix}-\mathbb{1}_2 & \\ & \mathbb{1}_2\end{pmatrix} \tag{S77}$$

in the Weyl representation. Recall that both $[\gamma^i, \gamma^j]$ and $[\gamma^i, \gamma^0]$ have Pauli matrices on the diagonal, but the latter has them with opposite signs. The presence of the γ^5 essentially exchanges the two. It is now the $[\gamma^i, \gamma^0]$ that contribute in the nonrelativistic limit, giving a term proportional to $\delta(\mathbf{p}-\mathbf{p}')\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{E})\xi_{s'}$. Physically, this corresponds to giving the particle an <u>electric dipole moment</u>.

The presence of an electric dipole moment implies the violation of CP symmetry. Many past and ongoing experiments search for the electric dipole moment of the electron, which could arise from physics beyond the Standard Model, but so far none has been detected. The Standard Model generically predicts a

sizable electric dipole moment for the neutron, which should have been easily detected decades ago, but mysteriously none has been found.

The above two are the only terms with dimension 5, and are therefore the simplest non-minimal couplings one could consider. Next, let's consider some dimension 6 terms.

c) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\Psi\,\partial^\nu F_{\mu\nu}$?

Solution: This is just like the coupling in problem 1, but we have replaced A_μ with $\partial^\nu F_{\mu\nu} = -J_\mu$. In the nonrelativistic limit, the leading effect is the one we found in problem 1(b), as this is the only one not suppressed by powers of v. By the same logic, the result is a term in H_I^{nr} proportional to $J^0(\mathbf{x}) = \rho(\mathbf{x})$, where ρ is the external charge density. That is, it has *no* effect, regardless of the fields at the particle's location, unless the particle is sitting right on top of some other charge.

This might seem rather mysterious, but it's perfectly comprehensible within classical electromagnetism. Consider a tiny spherical capacitor with places of charge $\pm q$. This charge configuration produces exactly zero electromagnetic field everywhere outside it, which means that it does not interact with external charges until they are right inside the capacitor, in which case they feel the potential inside.

If the particle already has a net charge, then superposing the above configuration spreads the charge out in space. For this reason, this term is said to produce a "charge radius". The charge radius is one way to quantify the size of particles like the proton. (The "proton radius puzzle" is the fact that two independent measurements of this quantity seemed to give different results.) When we say the electron is a pointlike particle, we mean that it has no charge radius, as far as we've detected.

d) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\gamma^5\Psi\,\partial^\nu F_{\mu\nu}$?

Solution: In this case the leading term in the nonrelativistic limit is that of 1(d), but with $\mathbf{B}$ replaced with $\mathbf{J}$, corresponding to an interaction of the form $\mathbf{S}\cdot\mathbf{J}(\mathbf{x})$. In other words, we have an interaction with currents directly on top of the particle.

To physically interpret this, imagine the particle is a tiny toroidal solenoid with its axis of symmetry aligned with $\mathbf{S}$. Such a configuration produces zero magnetic field outside itself, but the magnetic field affects currents that pass right over the toroid. When the current flows parallel to $\mathbf{S}$, it gets pushed directly towards (or away from) the toroid's axis, while a current flowing perpendicular to $\mathbf{S}$ experiences no net force; this therefore reproduces the $\mathbf{S}\cdot\mathbf{J}$ potential.

Depending on the field of physics, such a current configuration is called an anapole moment, or a polar toroidal dipole moment. You have probably not seen such a thing in electromagnetism classes, because it doesn't show up in the multipole expansion – it is a near field effect, not a far field effect. But anapole moments are important observables in nuclear physics.

One can keep going. The anapole moment produces a localized parity-violating magnetic field, while the so-called Schiff moment, also important in nuclear physics, produces a localized parity-violating electric field. It corresponds to a term in H_I^{nr} proportional to $\mathbf{S}\cdot\nabla\rho$.