

This problem set guides you through the calculation of loop diagrams and their application to the renormalization of QED. Unlike previous problem sets, it is ordered sequentially: each problem is easiest to approach after completing the previous problems. For reference, you may find it useful to consult sections 6.3 and 7.5 of Peskin and Schroeder, but this problem set is self-contained and its results are somewhat more general.

We will use dimensional regularization with $d = 4 - 2\epsilon$. To avoid confusion, we will rename the $i\epsilon$ in the Feynman propagator to $i0$ in this problem set.

1. $e^+e^- \rightarrow \mu^+\mu^-$ at one loop. (8 points)

In the previous problem set, you considered the leading order (or “tree level”) contribution to the scattering matrix element for $e^+e^- \rightarrow \mu^+\mu^-$,

$$\mathcal{M}^{\text{LO}} = \begin{array}{c} \text{Feynman diagram for LO process} \\ \text{with external lines labeled by momenta and times} \end{array} \quad (1)$$

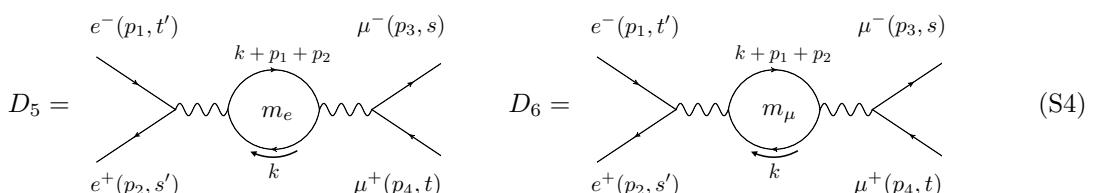
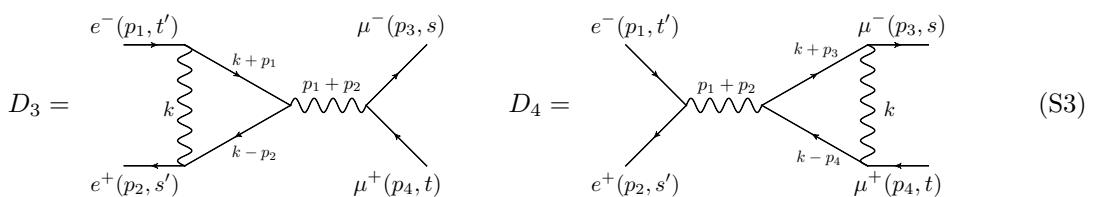
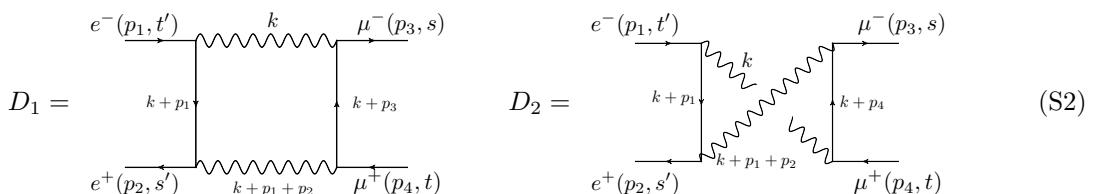
$$= \bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i0}\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \quad (2)$$

which is proportional to e^2 . The next-to-leading order contribution \mathcal{M}^{NLO} is of order e^4 , and contains Feynman diagrams with one loop.

- a) Draw all Feynman diagrams that contribute to \mathcal{M}^{NLO} .

Solution: The matrix element at next-to-leading order (NLO) is given by

$$M^{\text{NLO}} = D_1 + D_2 + D_3 + D_4 + D_5 + D_6, \quad (\text{S1})$$



b) Write down \mathcal{M}^{NLO} using the Feynman rules. You don't have to evaluate the loop integrals; this is just to show you examples of their typical form.

Solution:

$$D_1 = \int \frac{d^d k}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) \frac{i(\not{k} + \not{p}_3 + m_\mu)}{(k + p_3)^2 - m_\mu^2 + i0} (-ie\gamma^\nu) v_t(p_4) \frac{-i\eta_{\mu\sigma}}{k^2 + i0} \\ \times \bar{v}_{s'}(p_2)(-ie\gamma^\rho) \frac{i(\not{k} + \not{p}_1 + m_e)}{(k + p_1)^2 - m_e^2 + i0} (-ie\gamma^\sigma) u_{t'}(p_1) \frac{-i\eta_{\nu\rho}}{(k + p_1 + p_2)^2 + i0}. \quad (\text{S5})$$

$$D_2 = \int \frac{d^d k}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) \frac{i(\not{k} + \not{p}_4 + m_\mu)}{(k + p_4)^2 - m_\mu^2 + i0} (-ie\gamma^\nu) v_t(p_4) \frac{-i\eta_{\mu\sigma}}{k^2 + i0} \\ \times \bar{v}_{s'}(p_2)(-ie\gamma^\sigma) \frac{i(\not{k} + \not{p}_1 + m_e)}{(k + p_1)^2 - m_e^2 + i0} (-ie\gamma^\rho) u_{t'}(p_1) \frac{-i\eta_{\nu\rho}}{(k + p_1 + p_2)^2 + i0}. \quad (\text{S6})$$

$$D_3 = \int \frac{d^d k}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) v_t(p_4) \frac{-i\eta_{\mu\nu}}{(p_3 + p_4)^2 + i0} \frac{-i\eta_{\sigma\rho}}{k^2 + i0} \\ \times \bar{v}_{s'}(p_2)(-ie\gamma^\rho) \frac{i(\not{k} - \not{p}_2 + m_e)}{(k - p_2)^2 - m_e^2 + i0} (-ie\gamma^\nu) \frac{i(\not{k} + \not{p}_1 + m_e)}{(k + p_1)^2 - m_e^2 + i0} (-ie\gamma^\sigma) u_{t'}(p_1). \quad (\text{S7})$$

$$D_4 = \int \frac{d^d k}{(2\pi)^d} \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1) \frac{-i\eta_{\mu\nu}}{(p_1 + p_2)^2 + i0} \frac{-i\eta_{\sigma\rho}}{k^2 + i0} \\ \times \bar{u}_s(p_3)(-ie\gamma^\rho) \frac{i(\not{k} + \not{p}_3 + m_\mu)}{(k + p_3)^2 - m_\mu^2 + i0} (-ie\gamma^\nu) \frac{i(\not{k} - \not{p}_4 + m_\mu)}{(k - p_4)^2 - m_\mu^2 + i0} (-ie\gamma^\sigma) v_t(p_4). \quad (\text{S8})$$

$$D_5 = \int \frac{d^d k}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\nu) v_t(p_4) \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1) \\ \times \frac{-i\eta_{\mu\rho}}{(p_1 + p_2)^2 + i0} \frac{\text{tr} \left[(-ie\gamma^\rho) i(\not{k} + m_e) (-ie\gamma^\sigma) i(\not{k} + \not{p}_1 + \not{p}_2 + m_e) \right]}{(k^2 - m_e^2 + i0)(k + p_1 + p_2)^2 - m_e^2 + i0} \frac{-i\eta_{\sigma\nu}}{(p_1 + p_2)^2 + i0}. \quad (\text{S9})$$

$$D_6 = \int \frac{d^d k}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\nu) v_t(p_4) \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1) \\ \times \frac{-i\eta_{\mu\rho}}{(p_1 + p_2)^2 + i0} \frac{\text{tr} \left[(-ie\gamma^\rho) i(\not{k} + m_\mu) (-ie\gamma^\sigma) i(\not{k} + \not{p}_1 + \not{p}_2 + m_\mu) \right]}{(k^2 - m_\mu^2 + i0)(k + p_1 + p_2)^2 - m_\mu^2 + i0} \frac{-i\eta_{\sigma\nu}}{(p_1 + p_2)^2 + i0}. \quad (\text{S10})$$

If you collect all factors of i , e and (-1) , the common pre-factor is $i^8(-1)^6 e^4 = (4\pi\alpha)^2$.

2. Feynman parameters. (12 points)

As you saw in problem 1, loop integrals take a few generic forms. For simplicity, let's consider a scalar theory, which doesn't have nontrivial factors in the numerator of the loop integrand. Then some generic loop integrals are the "tadpole", "bubble", and "triangle",

$$I_T(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^a} \quad (3)$$

$$I_B(a, b; p^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^a ((k + p)^2)^b} \quad (4)$$

$$I_\Delta(a, b, c; p_1, p_2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^a ((k + p_1)^2)^b ((k - p_2)^2)^c} \quad (5)$$

named after the kinds of Feynman diagrams in which they appear. Here we have suppressed $i0$ terms, but you should keep in mind that they're implicitly there, which will be important in the last subpart. We regard a , b , and c as general real exponents, though in practice they will usually be positive integers. The dimension d is also a real parameter.

To compute the bubble and triangle integrals, it is helpful to introduce Feynman parameters, which combine the factors into the denominator into a power of a single quantity, like the tadpole integral. We will first have to build up some mathematical machinery.

a) The gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \quad (6)$$

and for positive integer n , satisfies $\Gamma(n) = (n - 1)!$. Show that for real ν ,

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx x^{\nu-1} e^{-xA}. \quad (7)$$

b) Show that for general ν_i ,

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma(\sum_{i=1}^n \nu_i)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i \right) \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{\prod_{i=1}^n x_i^{\nu_i-1}}{[\sum_{i=1}^n x_i A_i]^{\sum_{i=1}^n \nu_i}}. \quad (8)$$

Here, the x_i are called Feynman parameters. (Hint: you should *not* base your answer on the derivation in Peskin and Schroeder, which only works for integer ν . Instead, start from the result you proved in part (a).)

Solution: Using the result of part (a) n times yields

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty \prod_{i=1}^n dx_i x_i^{\nu_i-1} e^{-\sum_{i=1}^n x_i A_i}. \quad (\text{S11})$$

This is already pretty close to the desired answer, so we now introduce an identity

$$1 = \int_0^\infty dy \delta \left(y - \sum_{i=1}^n x_i \right) \quad (\text{S12})$$

which converts our result to

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy \prod_{i=1}^n dx_i x_i^{\nu_i-1} \delta \left(y - \sum_{i=1}^n x_i \right) e^{-\sum_{i=1}^n x_i A_i}. \quad (\text{S13})$$

Let us rescale all the x_i by y so they sum up to 1, as required in the final answer. The result is

$$\begin{aligned} \frac{1}{\prod_{i=1}^n A_i^{\nu_i}} &= \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy y^{\sum_{i=1}^n \nu_i} \prod_{i=1}^n dx_i x_i^{\nu_i-1} \delta \left(y - y \sum_{i=1}^n x_i \right) e^{-y \sum_{i=1}^n x_i A_i} \\ &= \frac{1}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^\infty dy y^{-1+\sum_{i=1}^n \nu_i} \prod_{i=1}^n dx_i x_i^{\nu_i-1} \delta \left(1 - \sum_{i=1}^n x_i \right) e^{-y \sum_{i=1}^n x_i A_i} \\ &= \frac{\Gamma(\sum_{i=1}^n \nu_i)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \prod_{i=1}^n dx_i x_i^{\nu_i-1} \delta \left(1 - \sum_{i=1}^n x_i \right) \frac{1}{[\sum_{i=1}^n x_i A_i]^{\sum_{i=1}^n \nu_i}} \end{aligned} \quad (\text{S14})$$

as desired, where we performed the y integral using (7).

- c) Apply (8) to $I_B(a, b; p^2)$ to get an integrand whose denominator contains a single quantity raised to the $a + b$ power.

Solution:

$$\begin{aligned} I_B(a, b; p^2) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} dx_2 x_2^{b-1} \frac{\delta(1-x_1-x_2)}{[k^2 x_1 + (k+p)^2 x_2]^{a+b}} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} dx_2 x_2^{b-1} \frac{\delta(1-x_1-x_2)}{[k^2(x_1+x_2) + (2kp+p^2)x_2]^{a+b}} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} (1-x_1)^{b-1} \frac{1}{[k^2 + (2kp+p^2)(1-x_1)]^{a+b}}. \end{aligned} \quad (\text{S15})$$

- d) Complete the evaluation of $I_B(a, b; p^2)$ by Wick rotating to Euclidean signature and performing all the remaining integrals. As a hint, the Euclidean integration measure is $d^d k_E = |k_E|^{d-1} d|k_E| d\Omega_d$, where the surface area of a unit sphere in d dimensions is

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (9)$$

As a second hint, you can use the result

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx x^{a-1} (1-x)^{b-1}. \quad (10)$$

To check your answer, if you set $d = 4 - 2\epsilon$ then you should find

$$I_B(1, 1; p^2) = \left(\frac{-p^2 - i0}{4\pi} \right)^{-\epsilon} \frac{i\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(4\pi)^2(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}. \quad (11)$$

Solution: Next, we perform the linear shift $k \rightarrow k - p(1-x_1)$ and find

$$I_B(a, b; p^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} (1-x_1)^{b-1} \frac{1}{[k^2 + p^2 x_1 (1-x_1)]^{a+b}}. \quad (\text{S16})$$

Notice, that we dropped the "+i0" Feynman prescription from the integral at the very beginning. We could have kept an infinitesimal mass term throughout. To track the infinitesimal imaginary part, we can simply associate it with the momentum, $p^2 + i0$. Next, we perform a Wick rotation $k^0 = ik_E^0$ and $\vec{k} = \vec{k}_E$.

$$\begin{aligned} I_B(a, b; p^2) &= i \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^d k_E}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} (1-x_1)^{b-1} \frac{1}{[-k_E^2 + p^2 x_1 (1-x_1)]^{a+b}} \\ &= \frac{i(-1)^{a+b}}{2(2\pi)^d} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^\infty d(k_E^2) \int d\Omega_d (k_E^2)^{\frac{d}{2}-1} \int_0^\infty dx_1 x_1^{a-1} (1-x_1)^{b-1} \frac{1}{[k_E^2 - p^2 x_1 (1-x_1)]^{a+b}} \\ &= \frac{i(-1)^{a+b}}{2(2\pi)^d} \frac{\Gamma(\frac{d}{2}) \Gamma(-a-b-\frac{d}{2})}{\Gamma(a)\Gamma(b)} \Omega_d \int_0^1 dx_1 x_1^{a-1} (1-x_1)^{b-1} ((-p^2 + i0)x_1(1-x_1))^{\frac{d}{2}-a-b} \\ &= \frac{i(-1)^{a+b} (-p^2 + i0)^{\frac{d}{2}-a-b}}{2(2\pi)^d} \frac{\Gamma(\frac{d}{2}) \Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)} \Omega_d \int_0^1 dx_1 x_1^{\frac{d}{2}-b-1} (1-x_1)^{\frac{d}{2}-a-1} \\ &= \frac{i(-1)^{a+b} (-p^2 + i0)^{\frac{d}{2}-a-b}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(a+b-\frac{d}{2})}{\Gamma(a)\Gamma(b)} \frac{\Gamma(\frac{d}{2}-a)}{\Gamma(d-a-b)}. \end{aligned} \quad (\text{S17})$$

Above we used the Euler β -function integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx x^{a-1} (1-x)^{b-1}. \quad (\text{S18})$$

3. Passarino–Veltman reduction. (10 points)

More generally, loop integrands will have momenta and other factors in the numerator. However, we can often use symmetry properties and a technique called Passarino–Veltman reduction to reduce them to the simpler loop integrands considered in problem 2. For example, consider the rank 2 tadpole integral

$$I_T^{\mu\nu}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^a}. \quad (12)$$

Because the right-hand side is a Lorentz tensor, the left-hand side must be as well. As there are no momenta in the integral, the only option is the metric, so we must have

$$I_T^{\mu\nu}(a; m^2) = \eta^{\mu\nu} A \quad (13)$$

for some Lorentz scalar A . Contracting both sides with the metric, we find

$$A = \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - m^2)^a} \quad (14)$$

$$= \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{(k^2 - m^2) + m^2}{(k^2 - m^2)^a}. \quad (15)$$

This is just two “scalar” tadpole integrals, as defined in (3), so we conclude

$$I_T^{\mu\nu}(a; m^2) = \frac{\eta^{\mu\nu}}{d} [I_T(a-1; m^2) + m^2 I_T(a; m^2)]. \quad (16)$$

a) Apply the same reasoning to the rank 3 and rank 4 tadpole integrals,

$$I_T^{\mu\nu\rho}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho}{(k^2 - m^2)^a}, \quad (17)$$

$$I_T^{\mu\nu\rho\sigma}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2 - m^2)^a}. \quad (18)$$

Solution: Again, only combinations of the metric tensor are available to us. There is no combination of metric tensors that would allow us to write down a tensor with 5 indices, so the first integral vanishes. (Another way of seeing this is that the integrand is odd under the transformation $k \rightarrow -k$ and the integral is consequently equal to minus itself and thus zero.)

The integrand with four Lorentz indices is symmetric under the exchange of any pair of indices, so we must have

$$I_T^{\mu\nu\rho\sigma}(a) = A (\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \quad (\text{S19})$$

Contracting with $\eta_{\mu\nu}\eta_{\rho\sigma}$ we find

$$I_T^{\mu\nu\rho\sigma}\eta_{\mu\nu}\eta_{\rho\sigma}(a) = A(d^2 + 2d) \quad (\text{S20})$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^2}{(k^2 - m^2)^a} \quad (\text{S21})$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{(k^2 - m^2 + m^2)^2}{(k^2 - m^2)^a} \quad (\text{S22})$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^{a-2}} + \int \frac{d^d k}{(2\pi)^d} \frac{2m^2}{(k^2 - m^2)^{a-1}} + \int \frac{d^d k}{(2\pi)^d} \frac{m^4}{(k^2 - m^2)^a} \quad (\text{S23})$$

from which we conclude

$$I_T^{\mu\nu\rho\sigma}(a; m^2) = \frac{\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}}{d^2 + 2d} (I_T(a-2; m^2) + 2m^2 I_T(a-1; m^2) + m^4 I_T(a; m^2)). \quad (\text{S24})$$

b) Similarly, the rank 1 and 2 bubble integrals can be written as

$$\begin{aligned} I_B^\mu(a, b; p^2) &= \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^a ((k+p)^2)^b} = p^\mu C, \\ I_B^{\mu\nu}(a, b; p^2) &= \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2)^a ((k+p)^2)^b} = \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) A_\perp + \frac{p^\mu p^\nu}{p^2} B_{||}. \end{aligned} \quad (19)$$

Express A_\perp and $B_{||}$ and C in terms of p^2 and the scalar bubble integral (4).

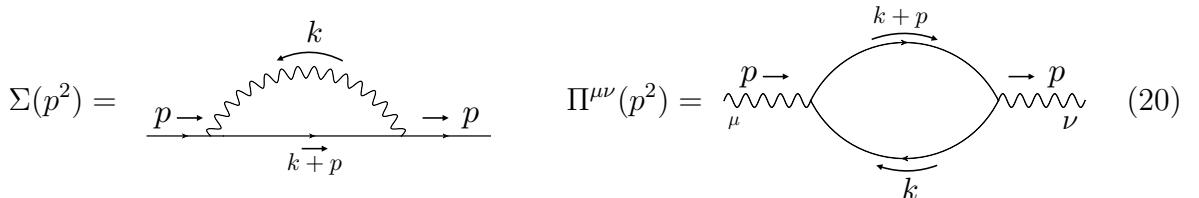
Solution:

$$\begin{aligned} I_B^\mu(a, b; p^2) p_\mu &= p^2 C = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{2kp}{(k^2)^a ((k+p)^2)^b} \\ C &= \frac{1}{2p^2} \int \frac{d^d k}{(2\pi)^d} \frac{(k+p)^2 - k^2 - p^2}{(k^2)^a ((k+p)^2)^b} \\ &= \frac{1}{2p^2} [I_B(a, b-1; p^2) - I_B(a-1, b; p^2) - p^2 I_B(a, b; p^2)]. \end{aligned} \quad (\text{S25})$$

$$\begin{aligned} I_B^{\mu\nu}(a, b; p^2) \eta_{\mu\nu} &= (d-1)A_\perp + B_{||} = I_B(a-1, b), \\ I_B^{\mu\nu}(a, b; p^2) \frac{p^\mu p^\nu}{p^2} &= B_{||} = \frac{1}{4p^2} \int \frac{d^d k}{(2\pi)^d} \frac{(2kp)^2}{(k^2)^a ((k+p)^2)^b} \\ &= \frac{1}{4p^2} \int \frac{d^d k}{(2\pi)^d} \frac{((k+p)^2 - k^2 - p^2)^2}{(k^2)^a ((k+p)^2)^b} \\ &= \frac{1}{4p^2} [I_B(a-2, b; p^2) + I_B(a, b-2; p^2) + (p^2)^2 I_B(a, b; p^2) \\ &\quad - 2 I_B(a-1, b-1; p^2) - 2p^2 I_B(a-1, b; p^2) - 2p^2 I_B(a, b-1; p^2)] \\ A_\perp &= \frac{1}{d-1} [I_B(a-1, b; p^2) - B_{||}]. \end{aligned} \quad (\text{S26})$$

4. Self-energy corrections in QED. (10 points)

Loop corrections to the two-point correlation function play a key role in renormalization, and are often referred to as self-energy corrections. The one-loop self-energy corrections for the electron and photon in QED are given by the diagrams below.



Throughout this problem we will work in massless QED, $m_e = 0$.

- a)** Using the Feynman rules, write down $\Sigma(p^2)$ and $\Pi^{\mu\nu}(p^2)$. Since we are viewing these quantities as corrections to the electron and photon propagators, you should *not* include factors for the external legs, such as $u_s(p)$ or ϵ_μ , to get a scalar matrix element. Instead, Σ should be a 4×4 spinor matrix and $\Pi^{\mu\nu}$ a rank 2 Lorentz tensor.

Solution: Using the Feynman rules, we simply read off the answers,

$$\Sigma(p^2) = \int \frac{d^d k}{(2\pi)^d} (-ie)\gamma^\mu \frac{i(\not{k} + \not{p})}{(k+p)^2 + i0} (-ie)\gamma^\nu \frac{-ig_{\mu\nu}}{k^2 + i0} \quad (\text{S27})$$

$$= -4\pi\alpha \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu(\not{k} + \not{p})\gamma_\mu}{(k^2 + i0)((k+p)^2 + i0)}. \quad (\text{S28})$$

and

$$\Pi^{\mu\nu}(p^2) = - \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr} [(-ie\gamma^\mu)i(\not{k} + \not{p})(-ie\gamma^\nu)i\not{k}]}{(k^2 + i0)((k+p)^2 + i0)} \quad (\text{S29})$$

$$= -4\pi\alpha \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr} [\gamma^\mu(\not{k} + \not{p})\gamma^\nu\not{k}]}{(k^2 + i0)((k+p)^2 + i0)}. \quad (\text{S30})$$

- b) Evaluate the resulting loop integrals, expressing your final result in terms of p^2 and ϵ . (Hint: once you handle the gamma matrices, all of the integrals you get will be ones evaluated earlier in the problem set.)

Solution: To compute Σ , we first note that $\gamma^\mu(\not{k} + \not{p})\gamma_\mu = (2-d)(\not{k} + \not{p})$. We then find that

$$\begin{aligned} \Sigma(p^2) &= -4\pi\alpha(2-d) [\not{p}I_B(1,1) + \gamma_\mu I_B^\mu(1,1)] \\ &= -2\pi\alpha(2-d)\not{p}I_B(1,1) \\ &= \frac{\alpha}{\pi} \left(\frac{-p^2 - i0}{4\pi} \right)^{-\epsilon} \not{p} \frac{i(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{4(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}. \end{aligned} \quad (\text{S31})$$

To compute $\Pi^{\mu\nu}$ we first compute the trace in the numerator.

$$\begin{aligned} \text{tr} [\gamma^\mu(\not{k} + \not{p})\gamma^\nu\not{k}] &= 4(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu}(k^2 + k.p)) \\ &= 4(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu} \left(k^2 + \frac{1}{2}((k+p)^2 - k^2 - p^2) \right)). \end{aligned} \quad (\text{S32})$$

With this we find

$$\begin{aligned} \Pi^{\mu\nu}(p^2) &= -16\pi\alpha \left[2I_B^{\mu\nu}(1,1;p^2) + p^\mu I_B^\nu(1,1;p^2) + p^\nu I_B^\mu(1,1;p^2) + \eta^{\mu\nu} \frac{p^2}{2} I_B(1,1;p^2) \right] \\ &= -8\pi\alpha p^2 \frac{d-2}{d-1} I_B(1,1;p^2) \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \\ &= -\frac{\alpha}{\pi} \left(\frac{-p^2 - i0}{4\pi} \right)^{-\epsilon} \frac{ip^2(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(1-2\epsilon)(3-2\epsilon)\epsilon\Gamma(1-2\epsilon)} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right). \end{aligned} \quad (\text{S33})$$

- c) The Euler–Mascheroni constant is defined as

$$\gamma_E = - \int_0^\infty dx e^{-x} \log x \approx 0.577. \quad (21)$$

Show that

$$\Gamma(1-\epsilon) = 1 + \gamma_E\epsilon + O(\epsilon^2). \quad (22)$$

Solution: Using the definition,

$$\Gamma(1-\epsilon) = \int_0^\infty dx x^{-\epsilon} e^{-x} \quad (\text{S34})$$

$$= \int_0^\infty dx e^{-\epsilon \log x} e^{-x} \quad (\text{S35})$$

$$= \int_0^\infty dx (1 - \epsilon \log x + O(\epsilon^2)) e^{-x} \quad (\text{S36})$$

$$= 1 + \gamma_E\epsilon + O(\epsilon^2) \quad (\text{S37})$$

as desired.

- d) Using (22), expand your results from part (b), dropping terms that vanish as $\epsilon \rightarrow 0$.

Solution: Plugging in the above result and simplifying gives

$$\Sigma = i \frac{\alpha}{\pi} \left(\frac{-p^2 - i0}{4\pi} \right)^{-\epsilon} e^{-\epsilon\gamma_E} \not{p} \left(\frac{1}{4\epsilon} + \frac{1}{4} + \mathcal{O}(\epsilon) \right) \quad (\text{S38})$$

and

$$\Pi^{\mu\nu}(p^2) = i \frac{\alpha}{\pi} \left(\frac{-p^2 - i0}{4\pi} \right)^{-\epsilon} e^{-\epsilon\gamma_E} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left(-\frac{1}{3\epsilon} - \frac{5}{9} + \mathcal{O}(\epsilon) \right). \quad (\text{S39})$$

5. ★ The scalar triangle integral. (5 points)

In this optional problem, we consider a somewhat more difficult loop integral.

- a) Evaluate the scalar triangle integral $I_\Delta(a, b, c; p_1, p_2)$ when $p_1^2 = p_2^2 = 0$, but for general a , b , c , and d , and give your answer in terms of $s = (p_1 + p_2)^2$.

Solution: Introducing $s = (p_1 + p_2)^2$, Feynman parameters and shifting the loop momentum $k \rightarrow k + (1 - x_1 - x_2)p_2 - x_2 p_1$ we find

$$I_T(a, b, c; s) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx_1 x_1^{a-1} dx_2 x_2^{b-1} (1-x_1-x_2)^{c-1} \frac{1}{[k^2 + sx_2(1-x_1-x_2)]^{a+b+c}}. \quad (\text{S40})$$

Next, we perform the Wick rotation $k^0 = ik_E^0$ and integrate out k_E . We find

$$I_T(a, b, c; s) = i(-1)^{a+b+c} \pi^{d/2} (-s - i0)^{-a-b-c+\frac{d}{2}} \frac{\Gamma(a+b+c-\frac{d}{2})}{(2\pi)^d \Gamma(a)\Gamma(b)\Gamma(c)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \\ \times x_1^{a-1} x_2^{-a-c+\frac{d}{2}-1} (1-x_1-x_2)^{-a-b+\frac{d}{2}-1}. \quad (\text{S41})$$

Next, we perform the transformation $x_2 = (1-x_1)x_2$ and subsequently integrate out the parameter integrals over x_1 and x_2 . We find

$$I_T(a, b, c; s) = i(4\pi)^{-\frac{d}{2}} (-1)^{a+b+c} (-s - i0)^{-a-b-c+\frac{d}{2}} \frac{\Gamma(-a-b+\frac{d}{2}) \Gamma(-a-c+\frac{d}{2}) \Gamma(a+b+c-\frac{d}{2})}{\Gamma(b)\Gamma(c)\Gamma(-a-b-c+d)}. \quad (\text{S42})$$

- b) It turns out there is a simple relation between the triangle and bubble integrals,

$$I_\Delta(1, 1, 1; p_1, p_2) \Big|_{p_1^2 = p_2^2 = 0} = C I_B(1, 1; (p_1 + p_2)^2). \quad (23)$$

Find the coefficient C in terms of d and s .

Solution: By comparing our previous results, we conclude

$$C = -\frac{2(d-3)}{(d-4)s}. \quad (\text{S43})$$