Electromagnetism II: Electricity

Chapters 3 and 4 of Purcell cover the material presented here, as does chapter 6 of Wang and Ricardo, volume 2. Image charges are covered in more detail in section 3.2 of Griffiths. For an array of interesting physical examples, see chapters II-6 through II-9 of the Feynman lectures. There is a total of **77** points.

1 The Method of Images

Idea 1

The method of images can be used in some highly symmetric situations to compute the electric field in the vicinity of a conductor. Specifically, consider any configuration of static charges and take any equipotential surface containing some of the charges. Then the resulting field configuration outside that surface is the field configuration we would have if that surface bounded a conductor. This is simply because it has constant potential on the conductor surface, so it must be the right answer by the uniqueness theorem.

[4] **Problem 1.** The simplest application of the method of images is the case of a charge q a distance a from an infinite grounded conducting plane. This problem explores some of its subtleties, assuming you've already read the basic treatment in section 3.4 of Purcell.

(a) Find the force on the charge.

(b) Find the work needed to move the charge out to infinity. (Answer: $q^2/16\pi\epsilon_0 a$.)

(c) Find the total potential energy of the charges on the conducting plane, i.e. the potential energy associated only with their interaction with each other. (Answer: $q^2/16\pi\epsilon_0 a$.)

(d) Now suppose there is another parallel grounded conducting plane on the other end of the charge, a distance b away. How many image charges are needed now? Draw some of them.

(e) A conducting plane forces the electric field to be perpendicular to it. Suppose we somehow had a plane which made the electric field always *parallel* to it. (This assumption might sound unphysical, but it will actually be useful in later handouts.) Find the force on the charge.

Example 1

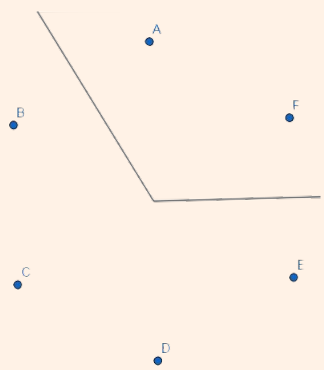
Two grounded conducting half-planes intersect, so that in cylindrical coordinates, the equations describing the planes are $\theta = 0$ and $\theta = \theta_p = \pi/2$. A charge q is placed somewhere between the planes. Can the method of images be used to find the force on the charge? What if $\theta_p = 2\pi/3$, or for general θ_p?

Solution

We can solve the first case with three image charges. Let the real charge q be at (x, y). Then we can reflect in the plane $\theta = 0$, adding an image charge $-q$ at $(x, -y)$ to satisfy its boundary condition. Then we can reflect both the real charge and this image charge in the plane $\theta = \pi/2$ to satisfy that plane's boundary condition, adding an image charge $-q$ at

$(-x, y)$ and an image charge q at $(-x, -y)$.

But when the other plane is at $\theta = 2\pi/3$, there is no configuration of image charges that works. For concreteness, let's suppose the real charge is at point A, on the y-axis.



Reflecting in the $\theta = 0$ plane forces us to have an image charge $-q$ at D, reflecting in the $\theta = 2\pi/3$ plane yields an image charge q at E, and reflecting in the $\theta = 0$ plane again yields a $-q$ charge at F, which is *real* since it's in the same region as A. But this isn't allowed: the point of image charges is to provide an easy way of calculating the effects of screening charges on conducting surfaces on a given set of *real* charges (i.e. the charge at A), so it's not legal to introduce *new* real charges in the process. We would get the same conclusion if we reflected about the planes in a different order – we always need a charge at F. More generally, the method of images works for this problem if and only if $\theta_p = \pi/n$ for integer n.

[4] **Problem 2.** In this problem you'll develop the method of images for spheres.

(a) A point charge $-q$ is located at $x = a$ and a point charge Q is located at $x = A$. Show that the locus of points with $\phi = 0$ is a circle in the xy plane, and hence a spherical shell in space.

(b) Show that the center of the sphere is at the origin provided that

$$a = \frac{r^2}{A}, \qquad -q = -\frac{Qr}{A}$$

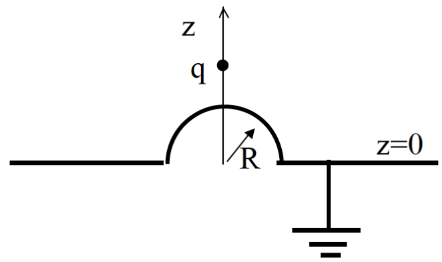
where r is the sphere's radius. These results will be used throughout the problem set.

(c) Now suppose a point charge Q is a distance b from the center of a spherical grounded conducting shell of radius r. Find the force on the charge, considering both the cases $b < r$ and $b > r$.

(d) The case $b < r$ is a bit confusing. On one hand, argue that the total charge is Q plus the image charge, and hence nonzero. On the other hand, argue that the total charge must be zero, by considering an appropriate Gaussian surface. One of these arguments is wrong – which one?

As an aside, the fundamental reason the method of images works for spheres is that electromagnetism has conformal symmetry, a symmetry under any local rescaling of space which preserves angles. (One example of a conformal transformation is inversion in Euclidean geometry.) The setup here is related to the conducting plane by such a transformation.

[2] **Problem 3.** An infinite grounded conducting plane at $z = 0$ is deformed with a hemispherical bump of radius R centered at the origin, as shown. A charge q is placed at $z = a$ as shown.



Can the method of images be used to find the potential in the region with the charge? If so, specify the image charges; if not, explain why not.

[2] **Problem 4** (Purcell 3.50)**.** A point charge q is located a distance $b > r$ from the center of a *nongrounded* conducting spherical shell of radius r, which also has charge q. When b is close to r, the charge is attracted to the shell because it induces negative charge; when b is large the charge is clearly repelled. Find the value of b so that the point charge is in equilibrium. (Hint: you should have to solve a difficult polynomial equation. You can either use a computer or calculator, or use the fact that it contains a factor of $1 - x - x^2$.)

[3] **Problem 5.** A neutral spherical conductor of radius R is placed in a uniform external field $\mathbf{E}_0$.

(a) Since electrostatic fields must vanish inside conductors, the surface charge on the conductor must conspire to create an opposing uniform field inside it. How exactly does this happen? Specifically, explicitly find $\sigma(\theta)$, the surface charge density as a function of the angle from $\mathbf{E}_0$. (Hint: we're already seen an example of a suitable charge density in **E1**.)

(b) Now let's consider the field created by this surface charge outside the sphere. We could integrate the answer to part (a), but it's even easier to use the method of images. Suppose that the original external field $\mathbf{E}_0$ was created by two very distant opposite point charges. Argue that the sphere picks up a dipole moment, and find its magnitude.

(c) What happens to the argument of part (b) if we instead suppose that $\mathbf{E}_0$ was created by a single very distant point charge?

This lessons of this problem will be useful in several later handouts.

[5] **Problem 6** (Purcell 3.45)**.** [A] Consider a point charge q located between two parallel infinite grounded conducting planes. The planes are a distance ℓ apart, and the point charge is a distance b from the left plane. The goal of this problem is to find the total charge induced on each plane.

(a) Argue that the total charge on each plane would not change if we replaced the point charge q with two point charges $q/2$, both a distance b from the left plane. By iterating this process, convert the point charge into a uniformly charged plane, and use this to get the answer.

(b) Alternatively, using image charges, show that the electric field on the inside surface of the left plane, perpendicular to the plane, at a point a distance r from the axis containing all the image charges, satisfies

$$4\pi\epsilon_0 E_\perp = \sum_{n=-\infty}^{\infty} \frac{2q(2n\ell + b)}{((2n\ell + b)^2 + r^2)^{3/2}}.$$

(c) Since $\sigma = -\epsilon_0 E_\perp$, we can integrate both sides to find the total charge on the left plane. However, the integral of each term by itself is simply q, so the series doesn't converge. To get the result, do the following steps in this specific order: group the terms $\pm n$ together, then integrate them only out to a distance $R \gg b$, then sum over the values of $|n|$, then take the limit $R \to \infty$. You should get a finite result that matches that of part (a). As you'll probably see in the process, if you do the steps in any other order, you'll get a nonsensical answer.

Those concerned with mathematical rigor might be bothered by the many choices made in part (c). You might ask, couldn't we have gotten a different result by changing how we did the computation? In fact, by the Riemann rearrangement theorem, we could have gotten almost *any* result. But the way we did it is the physically correct way. It roughly sums the terms "in to out", which respects the fact that real plates are finite. Closely related ideas are used to "cancel infinities" in quantum field theory, in a process known as renormalization. We'll see another example in **X1**.

2 Capacitors

Idea 2

There are multiple definitions of capacitance. For a single, isolated conductor with charge Q, the self-capacitance is defined as

$$Q = C\phi$$

where ϕ is the potential difference between the conductor and infinity. But for a set of two isolated conductors with charges $\pm Q$, you can also define a "mutual" capacitance by

$$Q = C\phi$$

where ϕ is the potential difference between the two conductors. When someone talks about a "capacitance" without specification, such as in idea 4, they probably mean this latter one.

Idea 3

The definitions of C above are only useful when you have only one or two conductors in the problem, respectively. In a situation with more than two, it's very tricky to use the above definitions, because all the conductors will affect each other; even a neutral conductor will have an effect since there will be induced charges on its surface.

Instead, it's better to revert to more general principles. The underlying principle behind capacitance is linearity: by the principle of superposition, the charges are linearly related to the potentials. For multiple capacitors, the most general possible linear relation is

$$Q_i = \sum_j C_{ij}\phi_j$$

where conductor i has charge Q_i and potential ϕ_i, the potential is taken to be zero at infinity, and the C_{ij} are called general coefficients of capacitance, or in electrical engineering, the Maxwell capacitance matrix. Similarly, inverting this relation,

$$\phi_i = \sum_j p_{ij} Q_j$$

where the p_{ij} are called coefficients of potential. We then calculate these coefficients by considering some appropriately selected situations, and solving a system of equations.

In Olympiad physics, you'll almost never want to compute the C_{ij} or p_{ij} explicitly. Instead, the point here is that if you're given the charges and want the potentials, or vice versa, you can build up the answer you want using the principle of superposition, computing all the fields you need, e.g. using Gauss's law.

Remark

General capacitance coefficients are discussed further in section 3.6 of Purcell. One nontrivial fact is that $C_{ij} = C_{ji}$, which is proven by energy conservation in problem 3.64 of Purcell. Capacitance coefficients can be clunky to work with. For example, suppose you want to compute the familiar capacitance of a system of two conductors. By definition, we have

$$Q_1 = C_{11}\phi_1 + C_{12}\phi_2, \quad Q_2 = C_{21}\phi_1 + C_{22}\phi_2.$$

An ordinary two-plate capacitor corresponds to the special case of opposite charges on the plates, so we write $Q = Q_1 = -Q_2$. There is a potential difference V across the plates, so $\phi_1 = \phi_2 + V$, and plugging this in gives

$$Q = (C_{11} + C_{12})\phi_2 + C_{11}V, \quad -Q = (C_{22} + C_{21})\phi_2 + C_{21}V.$$

Eliminating ϕ_2 from the system of equations above, we find the familiar mutual capacitance

$$C = \frac{Q}{V} = \frac{C_{11}C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}}$$

where we used $C_{12} = C_{21}$. This is quite an inconvenient formula, so as a result we won't consider general capacitance coefficients any further, except briefly for practice in problem 8.

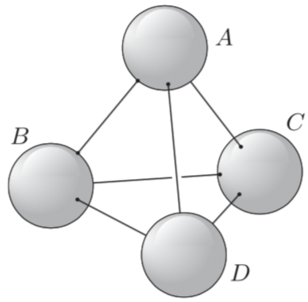
[2] **Problem 7** (Purcell 3.21)**.** Consider a capacitor made of four parallel plates with large area A, evenly spaced with small separation s. The first and third are connected by a wire, as are the second and fourth. What is the capacitance of the system?

[3] **Problem 8.** Consider two concentric spherical metal shells, with radii $a < b$.

(a) Compute their capacitance using Gauss's law.

(b) Compute their capacitance by computing the four capacitance coefficients, verifying that $C_{12} = C_{21}$ along the way, and using the result for C above.

[3] **Problem 9** (MPPP 152)**.** Four identical non-touching metal spheres are positioned at the vertices of a regular tetrahedron, as shown.



A charge $4q$ given to sphere A raises it to a potential V. Sphere A can also be raised to potential V if it and one of the other spheres are each charged with $3q$. What must be the size of equal charges given to A and two other spheres for the potential of A to again be raised to V? What if all four spheres are used?

[3] **Problem 10.** USAPhO 2008, problem A1.

Idea 4

A two-plate capacitor with voltage difference V and mutual capacitance C stores energy

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2.$$

Many circuits have multiple two-plate capacitors. In general, these need to be handled with the capacitance coefficients introduced in idea 3. But in practice, capacitors used in circuits are designed to produce fields confined within themselves, so that different capacitors don't interact with each other. In that case, we can just use mutual capacitance throughout, and C adds in parallel, while $1/C$ adds in series. (But this not work if, e.g. you put one capacitor inside another, in which case you should think about the charges and fields directly.)

[2] **Problem 11** (Purcell 3.24)**.** Some estimates involving capacitance.

(a) Estimate the capacitance of the Earth.

(b) Make a rough estimate of the capacitance of the human body.

(c) By shuffling over a nylon rug on a dry winter day, you can easily charge yourself up to a couple of kilovolts, as shown by the length of the spark when your hand comes too close to a grounded conductor. How much energy would be dissipated in such a spark?

[2] **Problem 12.** The total energy can also be found by integrating the electric field energy,

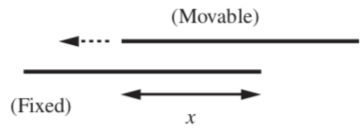
$$U = \frac{\epsilon_0}{2} \int E^2 \, dV.$$

(a) Show that this agrees with $U = CV^2/2$ for a parallel plate capacitor.

(b) Show that this agrees with $U = CV^2/2$ for a capacitor made of concentric spheres.

The general proof is more advanced, but if you're interested, one slick method is given in problem 1.33 of Purcell.

[3] **Problem 13** (Purcell 3.26). A parallel-plate capacitor consists of a fixed plate and a movable plate that is allowed to slide in the direction parallel to the plates. Let x be the distance of overlap.



The separation between the plates is fixed. Let $C(x)$ be the capacitance.

(a) Assume the plates are electrically isolated, so that their charges $\pm Q$ are constant. By differentiating the energy, find the leftward force on the movable plate in terms of Q and $C(x)$.

(b) Now assume the plates are connected to a battery, so that their potential difference ϕ is held constant. Find the leftward force on the movable plate, in terms of ϕ and $C(x)$.

(c) If the movable plate is held in place, the two answers above should be equal because nothing is moving. Verify that this is the case, being careful with signs.

(d) In terms of electric fields, why is there a force on the movable plate? Does the effect invoked in the answer to this part change the conclusion of parts (a) through (c) at all?

Idea 5: Dielectrics

While we're on the subject of capacitors, it's useful to introduce dielectrics. A dielectric is an insulator which polarizes in the presence of an electric field, with positive charges displaced slightly along the field. The resulting electric dipoles distributed throughout the material in turn create a field that tends to weaken the original applied field within the material.

Each part of a dielectric polarizes based on the local electric field, but that electric field depends on the applied field, and the polarization of *every other* piece of the dielectric. Thus, solving for the electric field for a general dielectric geometry is very difficult, and usually not possible in closed form, just like how it's usually not possible to solve for the field of a charged conductor. In Olympiad physics, you will almost always consider highly symmetric situations, where a dielectric simply reduces the applied electric field everywhere inside by a factor of κ, called the dielectric constant. (We'll consider some trickier situations in **E8**.)

Consider a parallel plate capacitor with charge $\pm Q$ on each plate. If a dielectric is inserted with the charge kept the same, then the field inside is reduced by a factor of κ. Thus, the capacitance $C = Q/V$ increases by a factor of κ. Dielectrics may increase the amount of energy that can be stored in a capacitor, which is typically limited by the voltage V_0 where electrical breakdown occurs. So if V_0 stays the same, the maximal stored energy $U = CV_0^2/2$ goes up by a factor of κ.

Plugging in the definition of C, this result implies that the energy density in the capacitor is $\kappa\epsilon_0 E^2$. But we showed in **E1** that the energy density of the electric field is only $\epsilon_0 E^2$. The extra energy is stored in the dielectric material itself: it takes energy to separate positive and negative charges within the dielectric, as if we were stretching many microscopic springs. This potential energy is released when the capacitor is discharged.

3 Tricky Problems

Example 2: PPP 151

A closed body with conducting surface F has self-capacitance C. The surface is now dented so that the new surface F^* is entirely inside F. Prove that the capacitance has decreased.

Solution

The energy stored in the capacitor is $U = Q^2/2C$. Therefore, if we give the capacitor a fixed charge Q, proving that F^* has lower C is equivalent to showing that it takes positive work to dent the foil from F to F^*. It's cleaner to show the other direction, i.e. that starting from F^*, we can get to F while only lowering the energy.

Suppose without loss of generality that F is infinitesimally larger than F^*. (We can break any finite change into infinitesimal stages and repeat this argument.) We can go from F^* to F by just taking each charge on the surface and moving it outward until it hits F. This lowers the energy because the electric field is always directed outward, as we proved in **E1**.

At this point, the charges lie on F, but they don't have the right distribution, i.e. F is not an equipotential. Now we let the charges spontaneously redistribute themselves so that F is again an equipotential. This again lowers the energy, proving the desired result.

Example 3

Are there charge distributions that aren't spherically symmetric, but which produce an *exact* $\hat{\mathbf{r}}/r^2$ field outside of them?

Solution

If you know a bit about the multipole expansion, this might seem like a daunting question. To make the field exactly $\hat{\mathbf{r}}/r^2$, you need to make sure the charge distribution has no dipole moment, no quadrupole moment, no octupole moment, and so on to infinity, and it seems impossible to satisfy all of these constraints without spherical symmetry. But we have *already* seen an example of such a charge distribution earlier in the problem set!

Recall that when we treated the method of images for spheres, we found that in some situations, the complicated charge densities on conducting spheres were exactly the same as those produced by a fictitious image charge inside the sphere, and generally away from its center. If we place the origin at that image charge, then we have an example of a charge distribution that is perfectly $\hat{\mathbf{r}}/r^2$ far away, but which isn't spherically symmetric. (The general solution is given here.)

[2] **Problem 14.** Consider a set of n conducting, very large parallel plates, placed in zero external electric field. The plates are given charges Q_i. If the left ends of the plates are at locations x_i, and the plates have thickness d_i, what is the total charge on the left end of the leftmost plate, and the right end of the rightmost plate?

[2] **Problem 15** (Purcell 3.9)**.** A conducting spherical shell has charge Q and radius R_1. A larger concentric conducting spherical shell has charge $-Q$ and radius R_2.

(a) If the outer shell is grounded, explain why nothing happens to the charge on it.

(b) If instead the inner shell is grounded, e.g. by connecting it to ground by a very thin wire that passes through a very small hole in the outer shell, find its final charge.

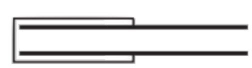
(c) It's not so clear why charge would leave the inner shell in part (b), thinking in terms of forces. A small bit of positive charge will certainly want to hop on the wire and follow the electric field across the gap to the larger shell. But when it gets to the larger shell, it seems like it has no reason to keep going to infinity, because the field is zero outside. And, even worse, the field will point *inward* once some positive charge has moved away from the shells. So it seems like the field will drag back any positive charge that has left. Does charge actually leave the inner shell? If so, what's wrong with the above reasoning?

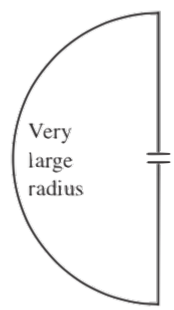
[2] **Problem 16.** The usual expression for the capacitance of a parallel-plate capacitor is $A\epsilon_0/d$. However, in reality the field within the capacitor is not perfectly uniform, and there are fringe fields outside. Is the true capacitance slightly higher or lower than $A\epsilon_0/d$?

[2] **Problem 17** (Purcell 4.16)**.** In a parallel plate capacitor, the quantity $\int \mathbf{E} \cdot d\mathbf{s}$ should be equal to V for any path that connects the two plates.

A charged capacitor can be discharged by attaching a wire to the external surfaces of the plates. No matter how one attaches the wire, $\int \mathbf{E} \cdot d\mathbf{s}$ along the wire should be equal to V. And as we've argued in **E2**, this is sufficient to cause charges to move along the wire, even if the electric field points in the "wrong" direction at some points along the wire, because the wire has negligible capacitance: charges within it move rigidly, each pushing the next one and pulling the previous one.

But it's puzzling how this works for a capacitor, because the electric field is supposed to be essentially zero just outside it. Consider two possible limiting cases for the wire's shape.





In each case, explain qualitatively how $\int \mathbf{E} \cdot d\mathbf{s}$ can be equal to V. In particular, how large are the contributions from the distinct segments of the wire (the horizontal and vertical parts in the first case, and the straight and curved parts in the second)?

[3] **Problem 18.** USAPhO 2022, problem A2. A computational problem involving surface tension.

Example 4

Find the leading interaction force between a dipole of dipole moment p and a grounded conducting sphere of radius r, separated by a distance $R \gg r$. What if the sphere is electrically neutral instead?

Solution

Place the origin at the center of the sphere and orient the z-axis to pass through the dipole. We can regard the dipole $p = qd$ as a combination of two charges

$$q \text{ at } z = R, \quad -q \text{ at } z = R + d$$

where d is very small. In the grounded case, this induces two image charges in the sphere,

$$\frac{qr}{R} \text{ at } z = \frac{r^2}{R}, \quad -\frac{qr}{R+d} \text{ at } z = \frac{r^2}{R+d}$$

approximately separated by r^2/R^2. We can now use Coulomb's law four times, but that's a bit tedious. Instead, decompose the image charges into a dipole moment and a net charge,

$$p' = \frac{pr^3}{R^3}, \quad Q' = \frac{qr}{R} - \frac{qr}{R+d} \approx \frac{pr}{R^2}.$$

We can place both of these at the origin, because this slight displacement will only affect the answer by subleading terms in r/R. Then the corresponding fields, far along the z-axis, are

$$E_{p'}(z) = \frac{2kpr^3}{R^3 z^3}, \quad E_{Q'}(z) = \frac{kpr}{R^2 z^2}.$$

The first term is negligible compared to the second, due to the many powers of R and z in the denominator. Thus, keeping only the second term, the force on the original dipole is

$$F = p \frac{d}{dz} E(z) \bigg|_{z=R} = -\frac{2kp^2 r}{R^5}$$

which falls off very quickly with distance. This derivation illustrates a common subtlety: it might not always be obvious how far to approximate. We threw away terms subleading in r/R, because we only wanted the leading contribution. But if we had applied that principle to the image charges at the first step, we would have thrown out the tiny net charge Q', which actually provides the dominant contribution to the force, because of how tiny p' is.

Now, the situation for a neutral sphere is completely different. By the logic of problem 4, there's a third image at the center of the sphere to enforce neutrality,

$$-\frac{pr}{R^2} \text{ at } z = 0.$$

The image charges can now be decomposed into a combination of two dipole moments. We already saw the first one p' above, while the second is, to leading order

$$p'' \approx \frac{pr}{R^2} \frac{r^2}{R} = \frac{pr^3}{R^3}$$

with the same magnitude and direction as p'. Thus, this system of image charges has approximate dipole moment $2p'$. The corresponding force is

$$F = p \frac{d}{dz} \frac{4\,kpr^3}{R^3 z^3} \bigg|_{z=R} = -\frac{12\,kp^2 r^3}{R^7}$$

which falls off even more quickly with distance. In this derivation, we didn't have to worry too much about getting p'' exactly right, because there was no net charge ("monopole") term that could've overwhelmed the dipole field, so all other field contributions are safely suppressed by more powers of r/R. (Of course, if p'' had come out pointing the opposite direction to p', so that the two almost cancelled, we would've had to be more careful.)

The lesson of this example is *not* to just use exact expressions and Taylor expand at the end. Here, that brute force approach would have required Taylor expanding six Coulomb's law forces out to order $1/R^7$, which is extraordinarily tedious. Instead, to approximate properly, we have to think carefully in every case. Incidentally, when applied to a polar and neutral nonpolar molecule, the $1/R^7$ force above is called the Debye force; it is one of the "van der Waals forces" which are often vaguely described in chemistry classes.

Example 5

Estimate the interaction force between a point charge q and a thin conducting rod of length ℓ, which is a distance $L \gg \ell$ from the charge and oriented along the separation between them.

Solution

The interaction occurs because the point charge induces negative charges on the near end of the rod, and positive charges on the far end. These charges are then acted on by the electric field of the point charge, causing a force.

To get a very crude estimate, let's just suppose that charge Q appears on the far end and charge $-Q$ appears on the near end. The resulting field produced in the middle is

$$E \sim \frac{kQ}{\ell^2}.$$

On the other hand, this needs to cancel a field from the point charge of

$$E \sim \frac{kq}{L^2}$$

which tells us that $Q \sim (\ell/L)^2 q$. The force on the induced charges is

$$F \sim kqQ\left(\frac{1}{(L+\ell)^2} - \frac{1}{L^2}\right) \sim -\frac{kqQ\ell}{L^3} \sim -\frac{kq^2\ell^3}{L^5}.$$

Again, the force is attractive, and falls off quickly with distance.

[3] **Problem 19** (Physics Cup 2017)**.** Estimate the interaction force between a point charge q and an infinitely thin circular neutral conducting disc of radius r if the charge is at the axis of the disc, and the distance between the disc and the charge is $L \gg r$.

[3] **Problem 20.** Consider two conducting spheres of radius r separated by a distance $a \gg r$, with total charges $\pm Q$. The spheres can be thought of as the two plates of a capacitor.

(a) Find a simple approximation for the capacitance C, valid when $a \gg r$.

In reality, the exact capacitance of this system can be written as an infinite series in r/a. Let's consider two ways of finding the corrections to the capacitance.

(b) By considering the energy $U = Q^2/2C$ of the system, find the first nontrivial correction to C.

(c) Alternatively, we can think about the charge distributions on the spheres. If we start with a "zeroth-order" uniform charge density on each sphere, it will induce a "first-order" image charge in the other sphere, which will in turn induce "second-order" image charges, and so on. We can then compute $C = Q/\Delta V$ by summing up all the image charges, and the total voltage difference they produce. Using this approach, find the first nontrivial correction to C.

(d) $\star$ It turns out that the quantity $1/C$ is a bit nicer than C. At what order in r/a does the *second* nontrivial correction to $1/C$ appear?

Example 6

Find the charge distribution on a conducting disc of radius R and total charge Q.

Solution

In general, there are very few situations where the charge distribution on a conductor can be found explicitly. As you've seen, some of the simplest examples can be solved with image charges. Some more complex, two-dimensional examples can be solved with a mathematical technique called conformal mapping. And this special example can be solved with a neat trick.

Consider a uniformly charged spherical shell centered on the origin, and consider a point P inside the shell, on the xy plane. The electric field at point P is zero, by the shell theorem. Recall that in the usual proof of the shell theorem, one draws two cones opening out of P in opposite directions. The charges contained in each cone produce canceling electric fields.

Now imagine shrinking the spherical shell towards the xy plane, so it becomes elliptical. The crucial insight is that the shell theorem argument above still works, for points on the xy plane. When we squash the shell all the way down to the xy plane, it becomes a disc, with zero electric field on it. This is thus a valid charge distribution for a disc-shaped conductor, and by the uniqueness theorem, it's the only one.

By keeping track of how much charge gets squashed to radius $[r, r + dr]$, we find $\sigma(r) \propto R/\sqrt{R^2 - r^2}$, and fixing the proportionality constant gives

$$\sigma(r) = \frac{Q}{4\pi R\sqrt{R^2 - r^2}}.$$

You can also show this by taking the $c, \epsilon \to 0$ limit of the "third shell theorem" in **M6**.

4 Electrical Conduction

We now leave the world of electrostatics and consider magnetostatics, the study of steady currents.

Idea 6

In a conductor with conductivity σ, the current density is

$$\mathbf{J} = \sigma \mathbf{E}.$$

Alternatively, $\mathbf{E} = \rho \mathbf{J}$ where ρ is the resistivity. The current and charge density satisfy

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

The current passing through a surface S at a given time is

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}.$$

Since $\mathbf{J} \propto \mathbf{E}$, we have Ohm's law $V = IR$, where V is the voltage drop across the resistor. The power dissipated in a resistor is $P = IV$. The resistance R adds in series, while $1/R$ adds in parallel.

[2] **Problem 21** (HRK)**.** A battery causes a current to run through a loop of wire.

(a) Suppose the wire makes a sharp corner. How do the charges know to turn around there?

(b) A copper wire with conductivity σ is joined to an iron wire with conductivity $\sigma' < \sigma$. For the current in both sections to be the same, the electric field in the iron wire must be higher. How does that happen?

In general, the surface charge distribution in a DC circuit can be quite complex; the aspects shown in these questions are just the beginning. For more about this, see this paper and this paper.

[1] **Problem 22.** A few decades ago, most light bulbs were incandescent. An incandescent light bulb is essentially just a resistor, which emits light when it gets hot. It is designed to be connected to a power supply of given voltage, in parallel with other bulbs. Now suppose a bulb marked "200 W" and a bulb marked "50 W" are accidentally connected in series. Which bulb is brigher?

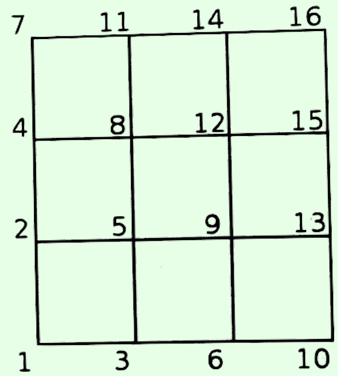
To warm up for DC circuits, we'll consider some resistor network problems.

Idea 7

If any two points in a resistor network are at the same potential, nothing will change if the two points are connected together and treated as one. More generally, the resistance of any resistor directly connecting the two points may be changed freely.

Example 7

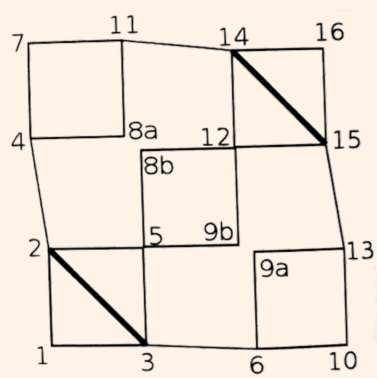
Consider the 3×3 grid below, where every edge is a resistor R.



Find the equivalent resistance between nodes 1 and 16.

Solution

By the above idea, we can short together nodes 2/3, and 14/15, by the diagonal symmetry of the network. Next, we can break nodes 8 and 9 into two pieces.



This is valid because the separated nodes $8a/8b$ and $9a/9b$ still have the same potential in the new network, by the diagonal symmetry. (This is using the above idea in reverse.) Now, the circuit has been reduced to combinations of series and parallel resistors. The resistance between 1 and 2/3 is $R/2$. The resistance between 2/3 and 14/15 is the combination of three networks in parallel, and finally the resistance of 14/15 and 16 is $R/2$. Thus,

$$R_{\text{eq}} = \left(\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right)^{-1} + \frac{1}{2}\right) R = \frac{13}{7} R.$$

You won't see any resistor problems as complicated as this one for the rest of the training, because they're kind of contrived; the point of this example was just to show multiple uses of symmetry techniques.

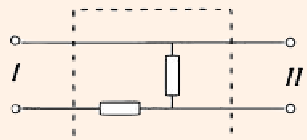
Example 8: PPP 23

A black box contains a resistor network and has two output terminals.

If a battery of voltage V is connected across the first terminal, the voltage across the second terminal is $V/2$. If a battery of voltage V is connected across the second terminal, the voltage across the first terminal is V. Find one possible configuration of the resistors inside the box.

Solution

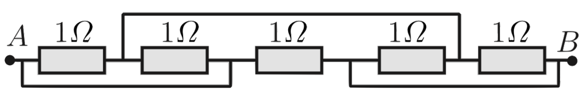
A simple configuration with two equal resistors works.



When a battery is connected across II, the horizontal resistor doesn't do anything. When a battery is connected across I, the two resistors comprise a voltage divider.

[2] **Problem 23.** USAPhO 2007, problem A1.

[2] **Problem 24** (IPhO 1996)**.** Consider the following resistor network.



Find the equivalent resistance between A and B.

[3] **Problem 25.** Consider a cube of side length L whose edges are resistors of resistance R.

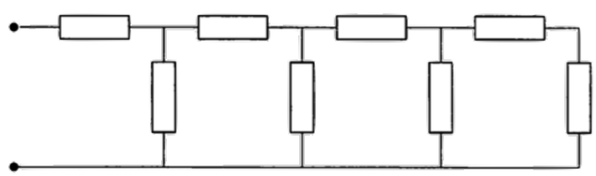
(a) Compute the resistance between two vertices a distance $\sqrt{3}L$ apart.

(b) Compute the resistance between two vertices a distance $\sqrt{2}L$ apart.

(c) Compute the resistance between two vertices a distance L apart.

(d) Generalize to vertices $\sqrt{n}L$ apart on an n-dimensional cube. (Give your answer in the form of a summation.)

[2] **Problem 26** (PPP 158)**.** Consider the circuit below, where every resistor is $1\,\Omega$.

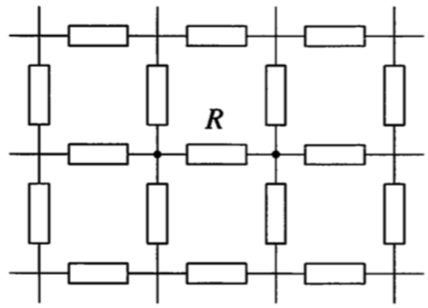


(a) Find the equivalence resistance between the input terminals.

(b) Do the same in the case where the chain is infinitely long.

[3] **Problem 27** (PPP 159-161)**.** Superposition can be a useful trick to analyze circuit networks.

(a) Consider an infinite two-dimensional grid of identical resistors R.



Find the equivalent resistance between two neighboring points by considering the superposition of a current I flowing into one point, and an equal current I flowing out the other.

(b) What would the equivalent resistance be if the resistor directly connecting the two neighboring points was removed?

(c) Now consider an icosahedron of identical resistors R. By superposing appropriate current distributions, find the equivalent resistance between two neighboring vertices.

Idea 8

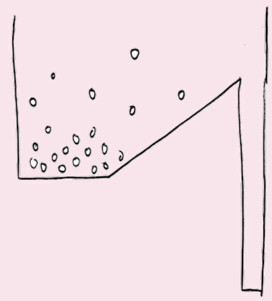
In a circuit of resistors and batteries, Kirchoff's loop rule states that the sum of the voltage drops around a loop is zero. Kirchoff's junction rule states that the net current flowing into a vertex is zero. (This is technically nonzero, because of the effect of problem 21, but negligible because wires have tiny capacitance.)

Remark

If the sum of the voltage drops around a loop is zero, then why would current ever want to flow? After all, if you had a circular tube of water, the water would never flow, because the net drop in height along the circle is zero. The reason current flows in circuits with batteries is that within the battery, charges are moved from lower to higher electric potential energy, just like how a pump could be used to move water upward to start a liquid circuit, by an "electromotive force".

But this immediately raises the question: what *is* this specific force? It can't be the electric force, because we just established that it's pointing the wrong way. It's not a magnetic effect. For some setups, it is literally a mechanical force like a pump: in the Van der Graaff generator, a motor drives the charges on a statically charged conveyor belt to higher potential. But that's not how batteries work.

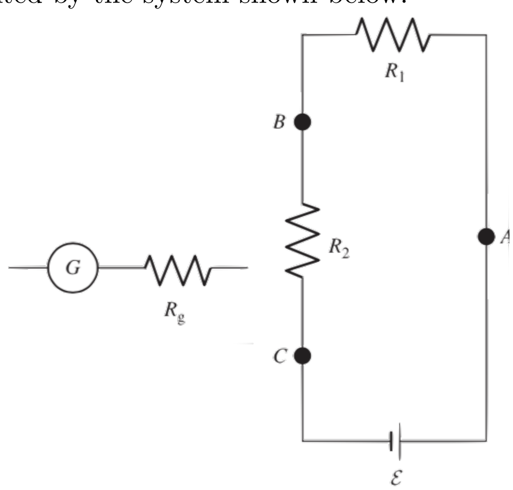
In a battery, there is no specific force pushing charges from low to high electric potential. Instead, the charges just jiggle around randomly, and the result emerges from the effects of their many collisions. To understand this, consider a gravitational analogy.



Consider an ideal gas at temperature T released in the trough shown above. The gas molecules will randomly collide, sometimes being propelled upward by chance. Sometimes, a gas molecule will climb the hill and fall into the deep hole, at which point it is unlikely to come out again. Thus, if the hole begins empty, it is energetically favorable for gas molecules to fill it. But there is no attractive force pulling molecules up along the slope! Gravity always points down; molecules go up the slope when they are randomly bounced that way.

This is essentially how the potential in an initially neutral battery is set up. The hole corresponds to the lower energy state an electron can reach inside the anode, but there is no long-range force pushing it there, just the average effect of random collisions.

[2] **Problem 28** (Purcell 4.10)**.** The basic ingredient in older voltmeters and ammeters is the galvanometer, a device to measure very small currents. (It works via magnetic effects, but the exact mechanism isn't important here.) Inherent in any galvanometer is some resistance R_g, so a physical galvanometer can be represented by the system shown below.



Consider a circuit such as the one shown, with all quantities unknown. We want to measure the current flowing across point A and the voltage difference between points B and C. Given a galvanometer with known R_g, and also a supply of known resistors (ranging from much smaller to much larger than R_g), how can you accomplish these two tasks? Explain how to construct your two devices (called an ammeter and voltmeter), and also how you should insert them in the given circuit. You will need to make sure that you (a) affect the given circuit as little as possible, and (b) don't destroy your galvanometer by passing more current through it than it can handle.

Remark

Occasionally, you might see Olympiad problems where a voltmeter is connected in series. The most common voltmeters are handheld digital multimeters, where the voltmeter setting presents a resistance of about $10\,\mathrm{M\Omega}$. Thus, for such problems, you should just treat the voltmeter like a high-resistance resistor.

Is this realistic? Well, it certainly happens every day, in almost every introductory physics lab in the world. But no professional would ever do this on purpose, because voltmeters aren't designed to be used this way. There is no guarantee that the resistance of the voltmeter is a constant. Instead, for most digital multimeters, there is a complex circuit inside that adjusts the internal resistance depending on the input and the configuration settings. You probably won't break the voltmeter when you put it in series, but you won't get reliable results either.

[2] **Problem 29.** USAPhO Quarterfinal 2009, problems 3 and 4.

[3] **Problem 30.** INPhO 2021, problem 1. A nice problem on practical circuit measurements. Note that the question statement is a bit vague. You are supposed to keep track of quantities of order R_A/R and R/R_V, but you are allowed to neglect quantities as small as R_A/R_V.