# Electromagnetism III: Magnetostatics

Chapters 4 and 6 of Purcell cover DC circuits and magnetostatics, as does chapter 5 of Griffiths. For advanced circuits techniques, see chapter 9 of Wang and Ricardo, volume 2. Chapter 5 of Purcell famously derives magnetic forces from Coulomb's law and relativity. It's beautiful, but not required to understand chapter 6; we will cover relativistic electromagnetism in depth in **R3**. For further discussion, see chapters II-12 through II-15 of the Feynman lectures. There is a total of **82** points.

## 1 Static DC Circuits

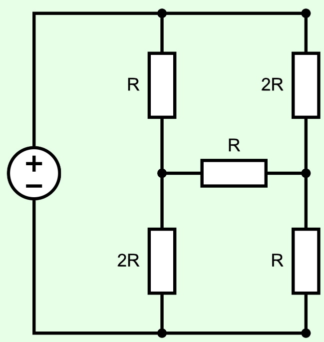
We continue with DC circuits, in more complex setups than in **E2**.

**Idea 1**

When analyzing circuits, it is sometimes useful to parametrize the currents in the circuits in terms of the current in each independent loop. This is typically more efficient, because it enforces Kirchoff's junction rule automatically, leading to fewer equations.

**Example 1: Imbalanced Wheatstone Bridge**

Find the current through the following circuit, if the battery has voltage $V$.




**Solution**

This circuit can't be simplified using series and parallel combinations, so instead we use Kirchoff's rules directly. From the diagram, we see the circuit has three loops. Let $I_1$ be the clockwise current on the left loop, $I_2$ be the clockwise current through the top-right loop, and $I_3$ be the clockwise current through the bottom-right loop. For instance, this means that the current flowing downward through the top-left resistor is $I_1 - I_2$.

The three Kirchoff's loop rule equations are

$$\begin{aligned} 3I_1R - I_2R - 2I_3R &= V, \\ 4I_2R - I_1R - I_3R &= 0, \\ 4I_3R - 2I_1R - I_2R &= 0. \end{aligned}$$

Adding the last two equations shows that

$$I_1 = I_2 + I_3$$

and plugging this back in shows that $3I_2 = 2I_3$, so we have

$$I_2 = \frac{2}{5} I_1, \quad I_3 = \frac{3}{5} I_1.$$

Since the answer to the question is just $I_1$, we can now plug this back into the first equation,

$$\frac{V}{R} = 3I_1 - I_2 - 2I_3 = \left(3 - \frac{2}{5} - \frac{6}{5}\right) I_1 = \frac{7}{5} I_1.$$

This gives the answer, $5V/7R$.

Incidentally, the Wheatstone bridge is a famous circuit with the same topology. We note that the current through the middle resistor is zero when the ratios between the top and bottom resistances match on both sides of it. Hence if three of these outer resistances are known, we can adjust one of them until the current through the middle resistor vanishes, thereby measuring the fourth resistor.

**Idea 2**

Since Kirchoff's loop equations are linear, currents and voltages in a DC circuit with multiple batteries can be found by superposing the currents and voltages due to each battery alone.

**Idea 3: Thevenin's Theorem and Norton's Theorem**

Consider any system of batteries and resistors, with two external terminals $A$ and $B$. Suppose that when a current $I$ is sent into $A$ and out of $B$, then a voltage difference $V = V_A - V_B$ appears. From an external standpoint, the function $V(I)$ is all we can measure.

Now, by the linearity of Kirchoff's rules, $V(I)$ is a linear function, so we can write

$$V(I) = V_{\text{eq}} + IR_{\text{eq}}.$$

In other words, $V(I)$ is exactly the same as if the entire system were a resistor $R_{\text{eq}}$ in series with a battery with emf $V_{\text{eq}}$ (with the positive end pointing towards $A$). This generalizes the idea of replacing a system of resistors with an equivalent resistance, and is known as Thevenin's theorem.

We can also flip this around. Note that $I(V)$ must also be a linear function, and we can write

$$I(V) = I_{\text{eq}} + \frac{V}{R_{\text{eq}}}.$$

This is precisely the $I(V)$ of an ideal current source $I_{\text{eq}}$ (sending current towards $B$) in parallel with a resistor $R_{\text{eq}}$. (An ideal current source makes a fixed current flow through it, just like a battery creates a fixed voltage across it.) This is known as Norton's theorem.

Since these functions are inverses of each other, you can see that the $R_{\text{eq}}$'s in both equations above are the same (both are equal to the ordinary equivalent resistance), and $V_{\text{eq}} = -I_{\text{eq}} R_{\text{eq}}$.

**Example 2**

Consider some batteries connected in parallel, with emfs $\mathcal{E}_i$ and internal resistances $R_i$. What is the Thevenin equivalent of this circuit?

**Solution**

The equivalent resistance is simply

$$R_{\text{eq}} = \left( \sum_i \frac{1}{R_i} \right)^{-1}.$$

To infer $V_{\text{eq}}$, we just need one more $V(I)$ value. The most convenient is to set $V = 0$, shorting all of the batteries. Each battery alone would produce a current of $\mathcal{E}_i / R_i$, so

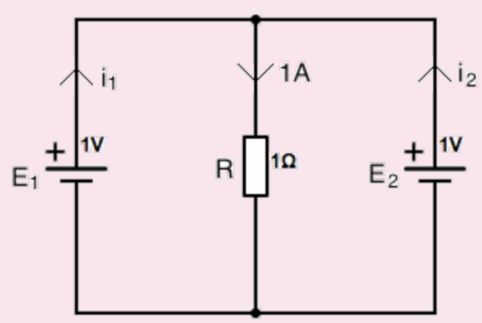
$$0 = V_{\text{eq}} + \left( \sum_i \frac{\mathcal{E}_i}{R_i} \right) R_{\text{eq}}.$$

Thus, we have

$$V_{\text{eq}} = \left( \sum_i \frac{\mathcal{E}_i}{R_i} \right) \left( \sum_j \frac{1}{R_j} \right)^{-1}.$$

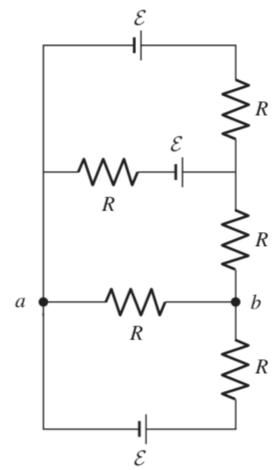
**Remark**

With ideal batteries, it's easy to set up circuits that don't make any sense.




For example, in the above circuit, Kirchoff's rules don't determine the currents; they only say that $i_1 + i_2 = 1\,\text{A}$. If the emfs of the batteries were different, the situation would be even worse: the equations would be contradictory, with no solution at all! In real life, this is avoided because all batteries have some internal resistance. Adding such a resistance to each battery, no matter how small, resolves the problem and gives a unique solution.

[2] **Problem 1** (Purcell 4.12)**.** Consider the circuit below.




(a) Find the potential difference between points $a$ and $b$.

(b) Find the equivalent Thevenin resistance and emf between points $a$ and $b$.

[2] **Problem 2** (Wang)**.** A circuit containing batteries and resistors has two terminals. When an ideal ammeter is connected between them, the reading is $I_1$. When a resistor $R$ is connected between them, the current through the resistor is $I_2$, in the same direction. What would be the reading $V$ of an ideal voltmeter connected between them?

[3] **Problem 3.** USAPhO 2015, problem A2.

Now we give a few problems on current flow through continuous objects. Fundamentally, all one needs for these problems is the definition $\mathbf{J} = \sigma \mathbf{E}$, and superposition.

**Example 3**

Consider two long, concentric cylindrical shells of radii $a < b$ and length $L$. The volume between the two shells is filled with material with conductivity $\sigma(r) = k/r$. What is the resistance between the shells, and the charge density?

**Solution**

To find the resistance, we compute the current $I$ when a voltage $V$ is applied between the shells. By symmetry, in the steady state the current density must be

$$\mathbf{J}(\mathbf{r}) = \frac{I}{2\pi r L}\,\hat{\mathbf{r}}.$$

On the other hand, we also know that

$$V = \int \mathbf{E} \cdot d\mathbf{r} = \int_a^b \frac{I}{2\pi r L \sigma}\,dr = \frac{I(b-a)}{2\pi k L}$$

from which we conclude

$$R = \frac{b-a}{2\pi k L}.$$

Note that the radial electric field between the shells is constant, so

$$\mathbf{E}(\mathbf{r}) = \frac{V}{b-a}\,\hat{\mathbf{r}}.$$

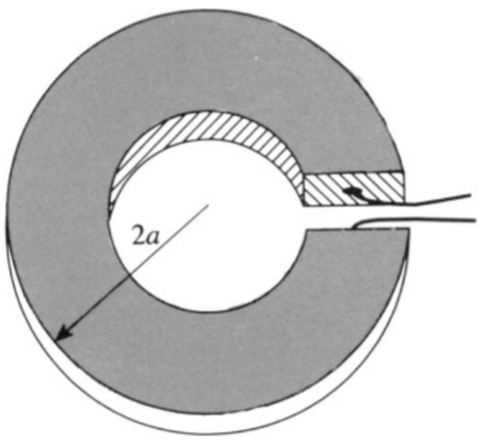
This means that in the steady state, there must be a nonzero charge density between the shells. (If there weren't, then we would have $E(r) \propto 1/r$, rather than a constant.)

To find the charge density explicitly, it's easiest to use Gauss's law in differential form in cylindrical coordinates. We use the form of the divergence derived in **E1**,

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \text{(other terms)} = \frac{1}{r}\frac{V}{b-a} = \frac{\rho}{\epsilon_0}$$

thus showing that the charge density is proportional to $1/r$. Of course, we could also get this result by applying Gauss's law in integral form, to concentric spheres.

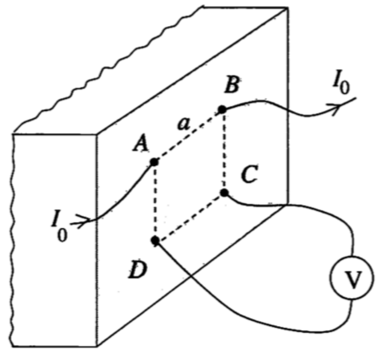
[2] **Problem 4** (Cahn)**.** A washer is made of a material of resistivity $\rho$. It has a square cross section of length $a$ on a side, and its outer radius is $2a$. A small slit is made on one side and wires are connected to the faces exposed.




Since the washer has an irregular shape, the current distribution inside it is complicated: it spreads out from the first wire, goes around the washer, and converges into the second wire. However, the situation is simpler if we glue a perfectly conducting square plate, of side length $a$, to each exposed face. Find the resistance in this case.

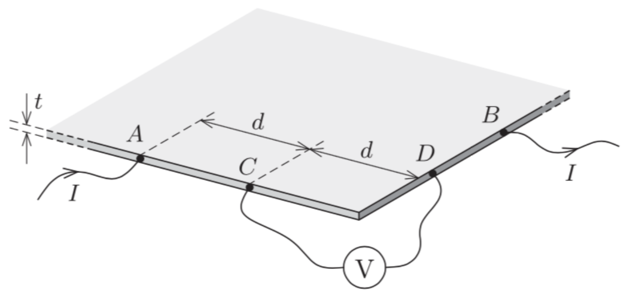
[3] **Problem 5** (BAUPC 1995)**.** An electrical signal can be transferred between two metallic objects buried in the ground, where the current passes through the Earth itself. Assume that these objects are spheres of radius $r$, separated by a horizontal distance $L \gg r$, and suppose both objects are buried a depth much greater than $L$ in the ground. If the Earth has uniform resistivity $\rho$, find the approximate resistance between the terminals. (Hint: consider the superposition principle.)

[3] **Problem 6** (PPP 162)**.** A plane divides space into two halves. One half is filled with a homogeneous conducting medium, and physicists work in the other. They mark the outline of a square of side $a$ on the plane and let a current $I_0$ in and out at two of its neighboring corners. Meanwhile, the measure the potential difference $V$ between the two other corners.




Find the resistivity $\rho$ of the medium.

[3] **Problem 7** (MPPP 174). We aim to measure the resistivity of the material of a large, thin, homogeneous square metal plate, of which only one corner is accessible. To do this, we chose points A, B, C and D on the side edges of the plate that form the corner.




Points A and B are both $2d$ from the corner, whereas C and D are each a distance $d$ from it. The length of the plate's sides is much greater than $d$, which, in turn, is much greater than the thickness $t$ of the plate. If a current $I$ enters the plate at point A, and leaves it at B, then the reading on a voltmeter connected between C and D is $V$. Find the resistivity $\rho$ of the plate material.

**Remark**

Setups like those in the previous two problems are commonly used to measure resistivities, but why do they use a complicated "four terminal" setup? Wouldn't it have been easier to just attach two terminals, send a current $I$ through them, and measure the voltage drop $V$? The problem with this is that it also picks up the resistance $R$ of the contacts between the terminals and the material, along with the resistances of the wires. By having a pair of terminals measure voltage alone, drawing negligible current, we avoid this problem.

[4] **Problem 8.** [A] This problem is just for fun; the techniques used here are too advanced to appear on Olympiads. We will prove Rayleigh's monotonicity law, which states that increasing the resistance of any part of a resistor network increases the equivalent resistance between any two points. This may seem obvious, but it's actually tricky to prove. The following is the slickest way.

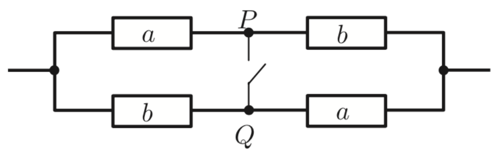
(a) Consider a graph of resistors, where a battery is attached across two of the vertices, fixing their voltages. Write an expression for the total power dissipated, assuming the voltages at each vertex are $V_i$ and the resistances are $R_{ij}$.

(b) The voltages $V_i$ at all the other vertices are determined by Kirchoff's rules. But suppose you didn't know that, or didn't want to set up those equations. Remarkably, it turns out that you can derive the exact same results by simply treating the voltages $V_i$ as free to vary, and setting them to minimize the total power dissipated! Show this result. (This is an example of a variational principle, like the principle of least action in mechanics.)

(c) For any network of resistors, show that $P = V^2/R$ when $V$ is the battery voltage applied across two vertices, $R$ is the equivalent resistance between them, and $P$ is the total power dissipated in the resistors. (This is intuitive, but it's worth showing in detail to assist with the next part.)

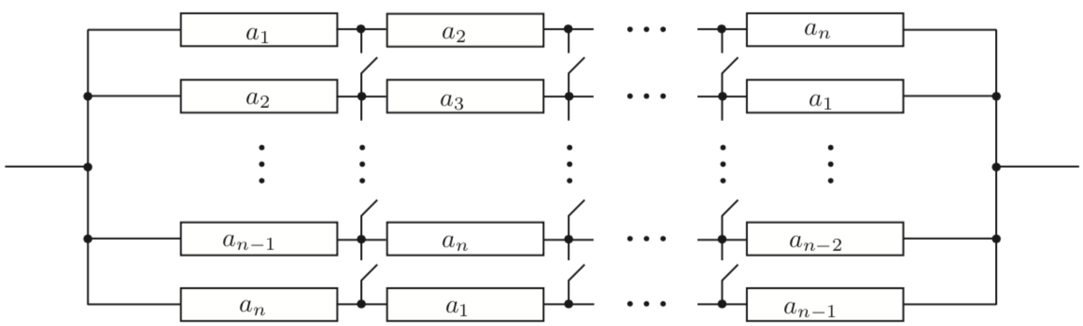
(d) By combining all of these results, prove Rayleigh's monotonicity law.

(e) We can use Rayleigh's monotonicity law to prove some mathematical results. Consider the resistor network shown below, where the variables label the resistances.




By considering the resistances before and after closing the switch $PQ$, show that the arithmetic mean of two numbers is at least the geometric mean.

(f) Consider the resistor network shown below.




By closing all the switches, show that the arithmetic mean of $n$ numbers is at least the harmonic mean.

**Remark**

You might think that Rayleigh's monotonicity law is too obvious to require a proof; if you decrease a resistance, how could the net resistance possibly go up? In fact, this kind of non-monotonicity occurs very often! For example, Braess's paradox is that fact that adding more roads can slow down traffic, even when the total number of cars stays the same. A U.S. Physics Team coach has argued that allowing more team strategies can make a basketball team score less. For more on this subject, see the paper *Paradoxical behaviour of mechanical and electrical networks* or this video.

**Remark**

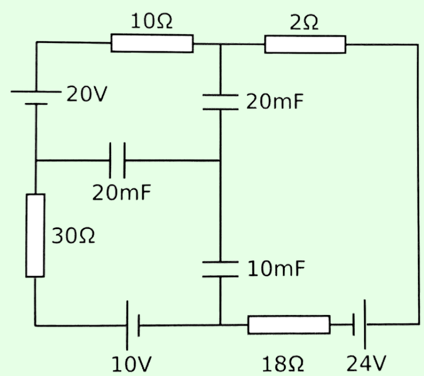
Circuit questions can get *absurdly* hard, but at some point they start being more about mathematical tricks than physics. As a result, I haven't included any such problems here; they tend not to appear on the USAPhO or IPhO, or in college physics, or in real life, or really anywhere besides a few competitions. On the other hand, you might find such questions fun! For some examples, see the Physics Cup problems 2013.6, 2017.2, 2018.1, and 2019.4.

## 2 RC Circuits

Next we'll briefly cover RC circuits, our first exposure to a situation genuinely changing in time.

**Example 4: CPhO**

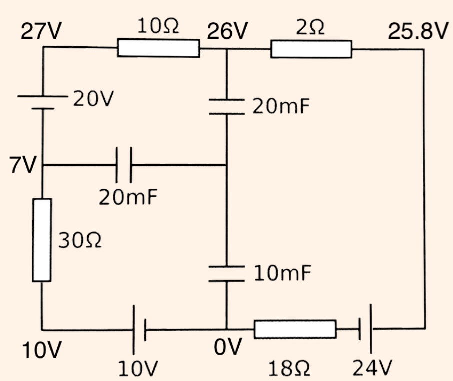
The capacitors in the circuit shown below were initially neutral. Then, the circuit is allowed to reach the steady state.




After a long time, what is the charge stored on the $10\,\mathrm{mF}$ capacitor?

**Solution**

After a long time, no current flows through the capacitors, so there is effectively a single loop in the circuit. It has a total resistance $60\,\Omega$ and a total emf $6\,\mathrm{V}$, so the current is $I = 0.1\,\mathrm{A}$. Using this, we can straightforwardly label the voltages everywhere on the outer loop.



To finish the problem, we need to know the voltage $V_0$ of the central node, so we need one more equation. That equation is charge conservation. We note that the central part of the circuit, containing the inner plates of the three capacitors, begins uncharged and is not directly connected to anything else, so it must remain uncharged. (Charge can't move through a capacitor!) Suppressing units, this means

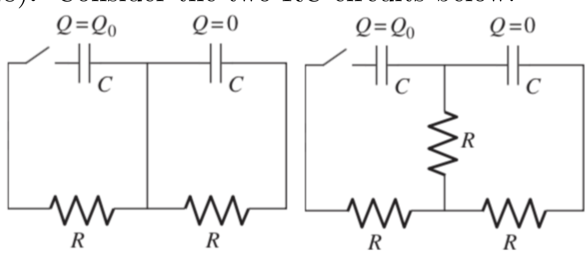
$$20(26 - V_0) + 20(7 - V_0) + 10(0 - V_0) = 0, \quad V_0 = \frac{66}{5}\,\mathrm{V}$$

from which we read off the answer,

$$Q = CV = 0.132\,\mathrm{C}.$$

[3] **Problem 9.** USAPhO 1997, problem A3.

[3] **Problem 10** (Purcell 4.18)**.** Consider the two RC circuits below.

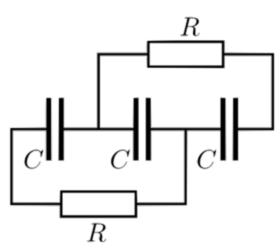



(a) The circuit shown below contains two identical capacitors and two identical resistors, with initial charges as shown above at left. If the switch is closed at $t = 0$, find the charges on the capacitors as functions of time.

(b) Now consider the same setup with an extra resistor, as shown above at right. Find the maximum charge that the right capacitor achieves. (Hint: the methods of **M4** can be useful.)

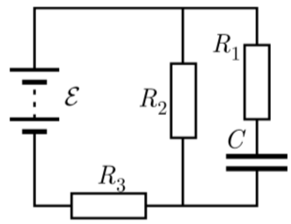
[3] **Problem 11.** USAPhO 2004, problem A1.

[3] **Problem 12** (Kalda)**.** Three identical capacitors are placed in series and charged with a battery of emf $\mathcal{E}$. Once they are fully charged, the battery is removed, and simultaneously two resistors are connected as shown.




Find the heat dissipated on each of the resistors after a long time.

[3] **Problem 13** (Kalda)**.** Find the time constant of the RC circuit shown below.




[3] **Problem 14** (MPPP 175/176)**.** A metal sphere of radius $R$ has charge $Q$ and hangs on an insulating cord. It slowly loses charge because air has a conductivity $\sigma$. In all cases, neglect any magnetic or radiation effects.

(a) Find the time $t$ for the charge to halve.

(b) By dimensional analysis, $t$ is independent of the radius $R$ of the metal sphere. More generally, show that $t$ takes the same value if the metal has *any* shape.

(c) Air has a conductivity of $\sigma \sim 10^{-13}\,\Omega^{-1}\text{m}^{-1}$, while water has a conductivity of $\sigma \sim 10^{-2}\,\Omega^{-1}\text{m}^{-1}$. About how long does the charge on an object last, if it is in air or water?

This problem generalizes USAPhO 2010, problem A2, which you can compare.

[5] **Problem 15.** IPhO 1993, problem 1. A really neat question with real-world relevance.

[5] **Problem 16.** IPhO 2007, problem "orange". A combination of mechanics and RC circuits.

## 3 Computing Magnetic Fields

**Idea 4**

The Biot–Savart law is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{s} \times \mathbf{r}}{r^3}.$$

As a consequence, we have Ampere's law,

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

as well as Gauss's law for magnetism,

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

**Idea 5**

The force on a stationary wire carrying current $I$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F} = I \int d\mathbf{s} \times \mathbf{B}.$$

The energy of a magnetic field is

$$U = \frac{1}{2\mu_0} \int B^2 \, dV.$$

The magnetic dipole moment of a planar current loop of area $A$ and current $I$ is $m = IA$, with $\mathbf{m}$ directed perpendicular to the loop by the right-hand rule.

**Idea 6: Magnetic Dipoles**

Far from a magnetic dipole with magnetic moment $m$, its magnetic field is just the same as the electric field of an electric dipole,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{\mu_0}{4\pi r^3}(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}).$$

As with the electric dipole field, you don't need to memorize this, but you should remember that it's proportional to the dipole moment, falls off as $1/r^3$, and be able to sketch it.

You should have already seen basic examples of using the Biot–Savart law in Halliday and Resnick, such as the field of a circular ring of current on its axis. We'll start with some problems that are similarly straightforward, but more technically complex.

[3] **Problem 17.** This is a key question which will help you understand idea 7. A spherical shell with radius $R$ and uniform surface charge density $\sigma$ spins with angular frequency $\omega$ about a diameter.

(a) Find the magnetic field at the sphere's center.

(b) Find the magnetic dipole moment of the sphere.

(c) It can be shown that (1) the magnetic field inside the sphere is uniform, and (2) the magnetic field outside the sphere is exactly that of a magnetic dipole. (It requires doing some obnoxious integrals, as can be seen in section 5.4 of Griffiths.) Using this information, make a qualitatively accurate sketch of the field.

(d) There is a closely related question in electrostatics: suppose we had two solid spheres of the same radius $R$, with volume charge densities $\pm\rho$, and the spheres were displaced by a tiny distance $d \ll R$. Qualitatively, what would the electric field of this setup look like, for $r < R$ and $r > R$? How does it differ from your answer to part (c)?

[2] **Problem 18** (Purcell 6.12)**.** A ring with radius $R$ carries a current $I$. Show that the magnetic field due to the ring, at a point in the plane of the ring, a distance $r$ from the center, is given by

$$B = \frac{\mu_0 I}{2\pi} \int_0^\pi \frac{(R - r\cos\theta)R \, d\theta}{(r^2 + R^2 - 2rR\cos\theta)^{3/2}}.$$

In the $r \gg R$ limit, show that

$$B \approx \frac{\mu_0}{4\pi}\frac{m}{r^3}$$

where $m = IA$ is the magnetic dipole moment of the ring, as expected from idea 6.

[3] **Problem 19** (Purcell 6.14)**.** Consider a square loop with current $I$ and side length $a$ centered at the origin, with sides parallel to the $x$ and $y$ axes. Show that the magnetic field at $r\hat{\mathbf{x}}$ is $B \approx (\mu_0/4\pi)(m/r^3)$ for $r \gg a$, as expected from idea 6. Be careful with factors of 2!

[3] **Problem 20.** USAPhO 2012, problem A3.

**Idea 7: Magnetic Monopoles**

Far away from the center of the dipole, the magnetic field of a magnetic dipole has the same form as the electric field of an electric dipole. Therefore, we can often replace a magnetic dipole $m$ with a fictitious pair of "magnetic charges" $\pm q_m$ separated by $d$, where $q_m d = m$. This is called a "Gilbert dipole", in contrast to a true "Amperian dipole".

This was the default way to think about magnets in the 1800s, but was largely removed from American textbooks in the 1950s because it's misleading in general: magnetic charges don't actually exist in magnets, and applying this analogy will give the wrong fields *inside* the dipole, as you saw in problem 17 and will see another way in problem 21. However, if we only care about the field outside the magnet, the analogy works, and it's often the fastest way to solve problems. We'll return to this idea in greater depth in **E8**.

[3] **Problem 21.** USAPhO 2015, problem B2. A key problem which illustrates idea 7.

We now give a few arguments for computing fields using symmetry.

**Example 5: PPP 31**

An electrically charged conducting sphere "pulses" radially, i.e. its radius changes periodically with a fixed amplitude. What is the net pattern of radiation from the sphere?

**Solution**

There is no radiation. By spherical symmetry, the magnetic field can only point radially. But then this would produce a magnetic flux through a Gaussian sphere centered around the pulsing sphere, which would violate Gauss's law for magnetism. So there is no magnetic field at all, and since radiation always needs both electric and magnetic fields (as you'll see in **E7**), there is no radiation at all. In fact, outside the sphere the electric field is always exactly equal to $Q/4\pi\epsilon_0 r^2$, in accordance with Coulomb's law.

**Example 6**

Find the magnetic field of an infinite cylindrical solenoid, of radius $R$ and $n$ turns per unit length, carrying current $I$.

**Solution**

Orient the solenoid along the vertical direction and use cylindrical coordinates. By symmetry, the field must be independent of $z$. Now consider the radial component of the magnetic field $B_r$. Turning the solenoid upside-down is equivalent to reversing the current. But the former does not flip $B_r$ while the latter does, so we must have $B_r = 0$.

Now, by rotational symmetry, the tangential component $B_\phi$ must be uniform. But then Ampere's law on any circular loop gives $B_\phi(2\pi r) = 0$, so we must have $B_\phi = 0$ as well.

The only thing left to consider is $B_z$. By applying Ampere's law to small vertical rectangles, we see that $B_z$ is constant unless that rectangle crosses the surface of the solenoid. Furthermore, $B_z$ must be zero far from the solenoid, so it must be zero everywhere outside the solenoid. Now, for a rectangle of height $h$ that does cross the surface, Ampere's law gives

$$\oint \mathbf{B} \cdot d\mathbf{s} = B_z^{\text{in}} h = \mu_0 I_{\text{enc}} = \mu_0 n I h$$

which tells us that $B_z^{\text{in}} = \mu_0 n I$.

**Example 7**

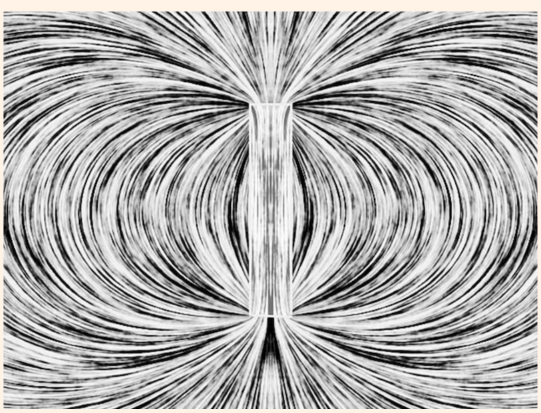
Now suppose the solenoid has finite length $L \gg R$. What do the fringe fields look like?

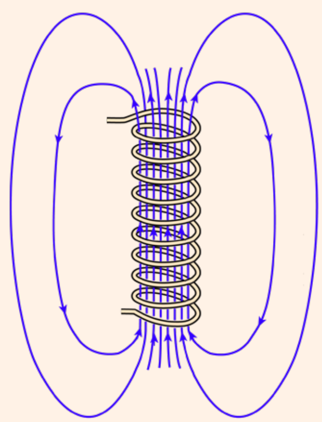
**Solution**

In principle we could solve for the exact fringe field by applying the Biot–Savart law to the solenoid wire, but that would be rather complicated. Instead, let's approximate the solenoid as a stack of $N = nL$ evenly spaced circular wire loops. Each one of these loops is a magnetic dipole $\mu = \pi R^2 I$, so the field of each loop well outside of it is just a dipole field.

Summing up all of these dipole fields is still complicated, so let's use idea 7. We can replace each wire loop with a pair of magnetic charges $\pm q_m$ separated by $d$, with the same magnetic dipole moment $\mu = q_m d$. If we choose $d = 1/n$, then the charges of adjacent dipoles cancel, leaving only charges $q_m = \pm n\mu = \pm \pi R^2 n I$ on the ends.

Thus, the fringe field of a solenoid, at distances much greater than $R$, looks like the electric field of two point charges! This is confirmed by a numeric calculation shown at left below.





This may come as a surprise to you if you've read basic, algebra-based introductory physics textbooks. Many of them contain hand-drawn diagrams like the one shown at right above, where all the magnetic flux comes neatly out the ends of the solenoids, in straight lines. In reality, the field sprays out almost spherically symmetrically from the end, with only half the flux actually going out through the end face, while the rest exits downward through the sides. (You will show this more directly with a slick argument in problem 23.)

We can also be more quantitative. Suppose the solenoid is vertical and centered at $z = 0$. Then the field at a radius $r$ from the solenoid axis, at $z = 0$, is

$$\mathbf{B}(r) = \mu_0 n I \hat{\mathbf{z}} \times \begin{cases} 1 & r < R \\ -2R^2/L^2 & R \ll r \ll L \\ -R^2 L/4r^3 & L \ll r \end{cases}$$

where the first line is the usual solenoid field, the second line is from applying Coulomb's law to our dipole analogy (which is only valid when $R \ll r$), and the third is from the dipole field of the two charges (only valid when $L \ll r$). As expected, in the limit $L \gg R$, the fringe field outside the solenoid is negligible. Another way of phrasing the result is that most of the upward flux through the solenoid returns through a downward field which mainly extends out to $r \sim L$. You can see all of these features in the accurate drawing above.

We can draw two lessons from this example. First, misleading diagrams are a common problem in introductory textbooks. A general rule is that the more basic a textbook is, the more pictures it'll have, but the less useful they'll be. Second, the analogy between Ampere and Gilbert dipoles is quite useful, and shows up frequently in tricky Olympiad problems.

**Remark: Real Solenoids**

Real solenoids are even more complicated. First, we didn't account for the discreteness of the wires. We just treated them as forming a uniform current per length $K = nI$, which is how we wrote $I_{\text{enc}} = nIh$. This is valid when you don't care about looking too closely, i.e. if your distance to any wire is much larger than the wire spacing $1/n$.

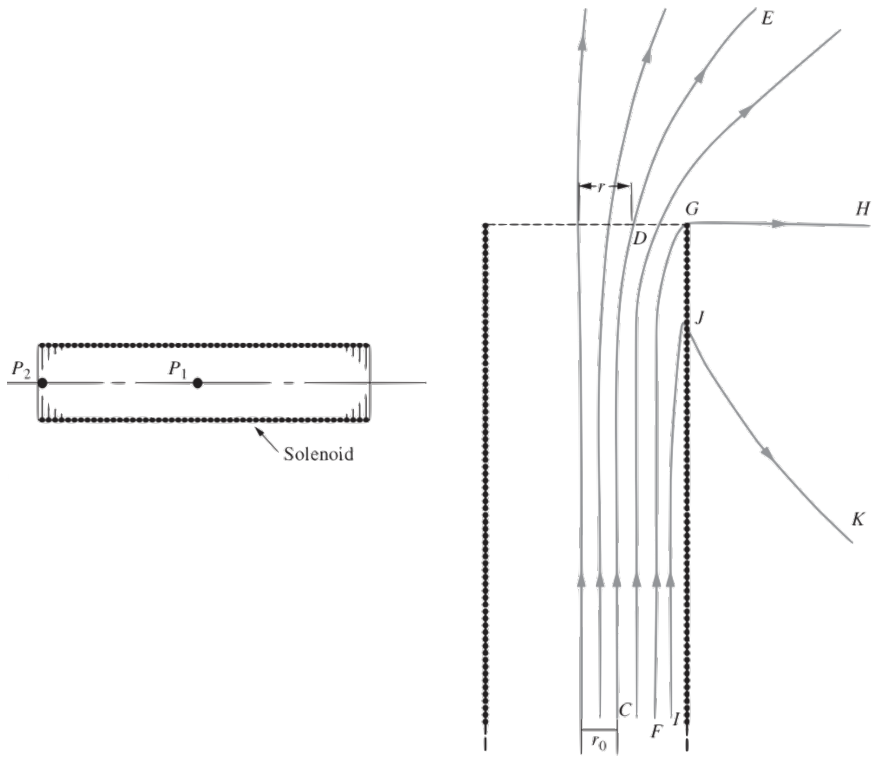
Second, the fact that solenoids are made by winding real wires means there is another contribution to the current, even in the limit $n \to \infty$. The wires are wound with a small slope, since a net current $I$ still has to move along the solenoid. Another way of saying this is that the current per length along the solenoid surface is $\mathbf{K} = nI\hat{\boldsymbol{\theta}} + (I/2\pi R)\hat{\mathbf{z}}$. This causes a tangential magnetic field $B_\phi = \mu_0 I/2\pi r$ outside the solenoid. Thus, in practice many solenoids are "counterwound": half the wires are wound evenly spaced going up the axis, and the other half are wound evenly spaced going back down the axis, which closes the loop and cancels this unwanted field.

[2] **Problem 22.** A toroidal solenoid is created by wrapping $N$ turns of wire around a torus with a rectangular cross section. The height of the torus is $h$, and the inner and outer radii are $a$ and $b$.

(a) In the ideal case, the magnetic field vanishes everywhere outside the toroid, and is purely tangential inside the toroid. Find the magnetic field inside the toroid.

(b) There is another small contribution to the magnetic field due to the winding effect mentioned above. Roughly what does the resulting extra magnetic field look like? If you didn't want this additional field, how would you design the solenoid to get rid of it?

[3] **Problem 23** (Purcell 6.63). A number of simple facts about the fields of solenoids can be found by using superposition. The idea is that two solenoids of the same diameter, and length $L$, if joined end to end, make a solenoid of length $2L$. Two semi-infinite solenoids butted together make an infinite solenoid, and so on.




Prove the following facts.

(a) In the finite-length solenoid shown at left above, the magnetic field on the axis at the point $P_2$ at one end is approximately half the field at the point $P_1$ in the center. (Is it slightly more than half, or slightly less than half?)

(b) In the semi-infinite solenoid shown at right above, the field line FGH, which passes through the very end of the winding, is a straight line from G out to infinity.

(c) The flux through the end face of the semi-infinite solenoid is half the flux through the coil at a large distance back in the interior.

(d) Any field line that is a distance $r_0$ from the axis far back in the interior of the coil exits from the end of the coil at a radius $r_1 = \sqrt{2}r_0$, assuming $\sqrt{2}r_0$ is less than the solenoid radius.

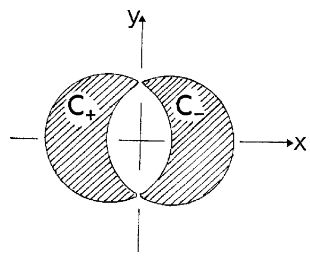
[3] **Problem 24** (MPPP 160). Two infinite parallel wires, a distance $d$ apart, carry electric currents along the $z$-axis with equal magnitudes but opposite directions. We can find the shape of the magnetic field lines with a neat trick, which only works for "two-dimensional" setups like this one, where the fields lie in the $xy$ plane and don't depend on $z$.

(a) Argue that if we rotated $\mathbf{B}$ by 90° in the $xy$ plane at each point, it would produce a valid electrostatic field $\mathbf{E}$. (Hint: consider rotating the $\mathbf{B}$ field of each wire individually.)

(b) Argue that the field lines of $\mathbf{B}$ are the same as the equipotentials of this artificial $\mathbf{E}$, and use this to find the field lines.

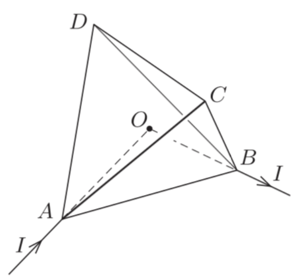
This trick is also useful for fluids in two dimensions, where it swaps vortices with sources and sinks.

[2] **Problem 25** (IPhO 1996). Two straight, long conductors $C_+$ and $C_-$, insulated from each other, carry current $I$ in the positive and the negative $\hat{\mathbf{z}}$ direction respectively. The cross sections of the conductors are circles of diameter $D$ in the $xy$ plane, with a distance $D/2$ between the centers.




The current in each conductor is uniformly distributed. Find the magnetic field in the space between the conductors.

[3] **Problem 26** (MPPP 157). A regular tetrahedron is made of a wire with constant resistance per unit length. A long, straight wire sends current $I$ into one vertex, and another long, straight wire removes it from another vertex, as shown.




Find the magnetic field at the center of the tetrahedron.

[5] **Problem 27.** APhO 2013, problem 1. A neat question on a cylindrical RC circuit that uses many of the techniques we've covered so far.