Mechanics I: Kinematics

See chapters 3 and 4 of Morin for material on solving differential equations. For general review on kinematics, see chapter 1 of Kleppner and Kolenkow. For fun, see chapters I-1 through I-8 of the Feynman lectures. There is a total of **92** points.

1 Motion in One Dimension

Example 1

When a projectile moves slowly through air, the drag is linear in the velocity, $F = -\alpha m v$. Find the velocity $v(t)$ of a projectile thrown upward at time $t = 0$ with speed v_0.

Solution

We write Newton's second law as

$$\frac{dv}{dt} = -g - \alpha v$$

and multiply through by dt. Integrating both sides from the initial condition to time t_f gives

$$\int_{v_0}^{v(t_f)} \frac{dv}{g + \alpha v} = -\int_0^{t_f} dt.$$

Performing the integrals gives

$$\frac{1}{\alpha} \log(g + \alpha v)\Big|_{v_0}^{v(t_f)} = -t_f.$$

Renaming t_f to t and solving for v yields

$$v(t) = e^{-\alpha t} v_0 + \frac{g}{\alpha}(e^{-\alpha t} - 1).$$

This renaming is necessary because we don't want to confuse t, the dummy variable that we integrating over, with t_f, the time at which we want to evaluate the velocity; t ranges from zero to t_f. Unfortunately, often people just call both of these t, so you need to watch out.

[2] **Problem 1.** Investigating some features of this solution.

(a) By using results from **P1**, verify that $v(t)$ makes sense for both small times and large times.

(b) If the projectile is then caught at the launch point, did it spend more time going up or down?

(c) Do you think the total time is longer or shorter than for a projectile without drag?

[3] **Problem 2.** Now assume quadratic drag, $F = -\alpha m v^2$, which applies for fast-moving projectiles.

(a) Integrate Newton's second law to get an implicit equation for $v(t)$ with the same initial conditions as above. That is, you don't need to solve for $v(t)$, as it'll just make things messy.

(b) Your equation will only be valid when the projectile is going up; explain why.

(c) Find $v(t)$ for an object released from rest at time $t = 0$. (Hint: if needed, look up some standard integrals involving hyperbolic trigonometric functions. But don't worry about memorizing the results, since in competitions, any nontrivial integral needed will usually be given to you.)

(d) Integrate your answer to part (c) with respect to time to find $y(t)$, and verify that the answer makes sense at both small and large times.

Some people only call this quadratic case drag; they call the linear case viscous resistance. This is because they behave fundamentally differently at the microscopic level, as we will explore in **M7**.

[3] **Problem 3.** A projectile of mass m is dropped from a height h above the ground. It falls and bounces elastically, experiencing the same quadratic drag as in problem 2. Find the maximum height to which it subsequently rises. (Hint: don't try to use your results from problem 2.)

Remark

How does the top speed v of a rowboat depend on the number N of rowers? A light, fast-moving rowboat experiences quadratic friction, so that the drag force on it is proportional to $v^2 A$, where A is the submerged cross-sectional area of the boat. A boat designed for N rowers will have a submerged volume $V \propto N$, and a streamlined shape so that $A \propto V^{2/3}$. Thus, the required power input is

$$P = Fv \propto v^3 N^{2/3}.$$

The power output by the rowers scale as N, and combining these results gives the amazingly weak dependence $v \propto N^{1/9}$, which agrees decently with Olympic rowing times.

Idea 1

An ordinary differential equation is any equation involving a quantity $x(t)$ and its derivatives. In introductory physics, we are usually concerned with a few very simple differential equations, with the following nice properties.

- The differential equation is at most second order, meaning it can contain x, its first derivative $\dot{x} = v$, and its second derivative $\ddot{x} = a$, but no higher derivatives. This implies the solution can be determined by an initial position and initial velocity. (We'll focus on second order differential equations for the rest of this section; most first order differential equations can simply be solved by separation and integration, as we've seen above.)

- The differential equation is linear, meaning that terms don't contain products of x, $\dot{x}$, and $\ddot{x}$. For example, a damped driven harmonic oscillator with time-dependent drag,

$$m\ddot{x} = -b(t)\dot{x} - kx + f(t)$$

is a second order linear differential equation. Solutions to such differential equations obey the superposition principle: if $x_1(t)$ and $x_2(t)$ are both solutions, so is $c_1 x_1(t) + c_2 x_2(t)$.

- The differential equation is homogeneous, meaning that each term is proportional to exactly one power of x or its derivatives. The above differential equation is not homogeneous, but it would be if we removed the driving $f(t)$.

- The differential equation is time-translation invariant, meaning that no functions of time appear except for x and its derivatives. The above equation isn't, but it would be if we set $f(t)$ and $b(t)$ to constants.

Idea 2

Linear, homogeneous, time-translation invariant differential equations are very special, and they can all be solved by the exact same method. First, note that we can promote $x(t)$ to a complex variable $\tilde{x}(t)$ and solve the differential equation over the complex numbers. As long as we have a complex solution, we can recover a real solution by taking the real part. Now, the method of solution, which works for *almost* all equations of this form, is to guess a complex exponential solution

$$\tilde{x}(t) = e^{i\omega t}.$$

Plugging this into the differential equation will yield the allowed values of ω, and the general solution can be found by superposing the complex exponentials.

Example 2

Solve the simple harmonic oscillator, $m\ddot{x} + kx = 0$, using the above principles.

Solution

First, we pass to a complex differential equation,

$$m\ddot{\tilde{x}} + k\tilde{x} = 0.$$

We guess $\tilde{x}(t) = e^{i\omega t}$. Plugging this in and using the chain rule gives

$$m(i\omega)^2 e^{i\omega t} + ke^{i\omega t} = 0$$

and canceling $e^{i\omega t}$ and solving gives two solutions,

$$\omega = \pm\omega_0, \quad \omega_0 = \sqrt{k/m}.$$

Since this a second-order linear differential equation, the general solution is given by the superposition of these two complex exponentials,

$$\tilde{x}(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

where A and B are general complex numbers. The real part of $\tilde{x}(t)$ satisfies the original real differential equation $ma + kx = 0$, and is

$$\operatorname{Re} x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where C and D are real numbers.

Remark

What's the meaning of promoting the real variable $x(t)$ to a complex variable $\tilde{x}(t)$? You can regard it as just a mathematical trick, which makes the solutions look a bit simpler. But for the specific case of the simple harmonic oscillator, there's a more satisfying explanation: $\tilde{x}(t)$ encodes the position of the particle in its real part, and the momentum of the particle (up to a scaling constant) in its imaginary part. As the system evolves, position rotates into momentum and vice versa. Energy conservation corresponds to the fact that the complex magnitude $|\tilde{x}|^2$ stays constant. We'll see a bit more of this perspective in **M4**.

[1] **Problem 4.** To make sure you know how to go from the complex solution to the real one, write C and D in terms of A and B.

[2] **Problem 5.** Now introduce a damping force and solve the differential equation for the damped harmonic oscillator, $m\ddot{x} + b\dot{x} + kx = 0$, using the same procedure, assuming b is small. (See section 4.3 of Morin if you have trouble with this. We'll consider this system in more detail in **M4**.)

[3] **Problem 6.** USAPhO 2012, problem B1.

[3] **Problem 7.** Above, we mentioned that guessing an exponential works *almost* all the time. The reason is because at the end of the day, the exponential cancels out and we're left with a polynomial in ω, which has just the right number of roots. But if there are repeated roots, there are fewer distinct solutions for ω, and hence not enough solutions.

(a) Write down a second order differential equation with a double root ω, and find its general solution. (Hint: to help find a good guess, consider the simple case $d^2x/dt^2 = 0$, where $\omega = 0$ is the double root. Then generalize your guess to nonzero ω and check that it works.)

(b) You should find that the solution qualitatively changes when you have an *exact* double root. However, in the limit where we have two roots that are very close together, $\omega \pm \Delta\omega$ with $\Delta\omega \ll \omega$, we should get approximately the same solution. Explicitly show how this works. When would you prefer to use either one?

(c) **[A]** Consider the most general n^{th} order, linear homogeneous time-translation invariant differential equation

$$\left(a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0\right) x = 0.$$

What does the general solution look like?

Remark

You might be wondering how to solve more general differential equations. In **M4**, we will consider three extensions of the above techniques. We'll use the idea of normal modes to solve systems of such differential equations, add driving forces to make the equations inhomogeneous, and use the adiabatic theorem to approximately solve non time-translation invariant equations where the coefficients change slowly in time.

Of course, this just scratches the surface of the subject, and solving more general differential equations can be orders of magnitude harder. We won't try to solve nonlinear differential equations, as there is no general technique for doing so, and the answer is often an obscure special function. (However, such equations will occasionally appear in later problems.) On the other hand, linear differential equations with general time-dependence are more approachable, and the following problem illustrates the most basic method for solving them.

[3] **Problem 8.** [A] Some linear, homogeneous, non time-translation invariant differential equations can be solved by simply guessing a power series. For this problem, don't worry about dimensional analysis; assume all variables have already been redefined to be dimensionless.

(a) As a warmup, consider the differential equation $\dot{x} = kx$ for constant k, which we already know how to solve. By plugging in the ansatz

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

with unknown constant coefficients a_n, find the solution with $x(0) = 1$.

(b) Now consider the non time-translation invariant differential equation

$$t^2 \ddot{x} + t\dot{x} + t^2 x = 0$$

which is called Bessel's differential equation of order zero. By using the same ansatz, find the unique solution with $x(0) = 1$ and $\dot{x}(0) = 0$.

2 Tricks

In this section we'll consider some kinematics problems that require cleverness, not computation.

Idea 3

Many problems can be solved by a clever choice of reference frame. It is often useful to go to the frame moving with one of the objects in the problem, or to go into a frame that makes the motion in the problem more symmetric. For the purposes of kinematics it can even be useful to use noninertial reference frames, such as a falling frame where projectiles don't accelerate, or a rotating frame, though this will introduce fictitious forces into the dynamics. It is also useful to tilt the coordinate axes to be parallel to various objects.

Example 3: $F = ma$ 2022 B4

A firework explodes, sending shells in all directions. Suppose the shells are all launched with the same speed, and ignore air resistance, but not gravity. What shape do the shells make?

Solution

In the absence of gravity, the shells would always form a sphere. Adding gravity simply shifts all of their locations downward by $gt^2/2$, so the shape is still always a sphere.

[1] **Problem 9** (KoMaL 2019)**.** A cannon A is at the edge of a cliff with a 800 m drop. Cannon B is on the ground below the cliff and 600 m horizontally away from it. Cannon A shoots a cannonball directly towards cannon B at 60 m/s. Cannon B shoots a cannonball directly towards cannon A at 40 m/s. Will the two cannonballs hit each other in midair?

[2] **Problem 10** (Wang)**.** Two particles are released in gravitational acceleration g with leftward and rightward speeds v_1 and v_2. Find the distance between them when their velocities are perpendicular.

[3] **Problem 11** (Kalda)**.** Two intersecting circles of radius r have centers a distance a apart. If one circle moves towards the other with speed v, what is the speed of one of the points of intersection?

[2] **Problem 12** (Kalda)**.** A mirror rotates about its center with angular speed ω. A stationary point source of light sits at a distance a from the rotation axis. What is the speed of its mirror image?

[2] **Problem 13** (Kalda)**.** Two circles of radius r intersect at the point O. One of the circles rotates about the point O with constant angular speed ω. The other point of intersection O' is originally a distance d from O. Find the speed of O' as a function of time.

Idea 4

To find the minimum value of some quantity, it's often useful to think about all possible values of that quantity. This can reveal a solution using geometry or symmetry.

[2] **Problem 14** (PPP 3)**.** A boat can travel a speed of 3 m/s on still water. A boatman wants to cross a river while covering the shortest possible distance.

(a) In what direction should he row if the speed of the water is 2 m/s?

(b) How about if it is 4 m/s?

Idea 5

In problems with friction, the best reference frame to use is almost always the frame of whatever is causing the friction.

[2] **Problem 15** (Kalda)**.** A block is pushed onto a conveyor belt. The belt is moving with speed 1 m/s, and the block's initial speed is 2 m/s, with initial velocity perpendicular to that of the belt. During the subsequent motion, what is the minimum speed of the block with respect to the ground?

Idea 6

For a variety of kinematics problems, it can be useful to think about the motion from a different perspective. For example, if your problem involves complicated accelerations, it can be useful to think in "velocity space", i.e. directly think about how the velocity vector evolves over time, and deal with the position later. Or, if your problem involves complicated processes occurring in time, it can be useful to think in "spacetime", meaning to visualize the process on a space where time is one of the axes. It can also be useful to parametrize motion in terms of quantities other than the usual Cartesian coordinates.

[2] **Problem 16** (Kalda)**.** A boy enters a patch of ice with a coefficient of friction μ with speed v. By running on the ice, the boy turns his velocity vector by 90° in the minimum possible time, so that his final speed is also v. What is the minimum possible time, and what kind of curve is the trajectory? Assume the normal force with the ice is constant.

[2] **Problem 17** (PPP 5)**.** Four snails travel in uniform, rectilinear motion on a plane. The velocities are chosen so that three snails never meet at once, and no two of the velocities are equal. Since time $t = -\infty$, five of the $\binom{4}{2}$ possible encounters have already occurred. Must the sixth also occur?

[2] **Problem 18.** Six bugs are placed at the vertices of a regular hexagon with side length s. At time $t = 0$ each bug starts moving directly towards the next with speed v. At what time do they collide?

Example 4

A rabbit begins at the origin, and the fox begins at the point $(0, -a)$. The rabbit begins running east, with a constant speed $v\hat{\mathbf{x}}$. At the same time, the fox begins chasing the rabbit, always moving towards it with speed v. After a long time, the rabbit and fox simply follow each other in a straight line, with a constant separation d. What is d?

Solution

This is the simplest example of a pursuit problem. Physicists and mathematicians have been posing them for centuries, though most are too mathematically involved for Olympiads.

Here, the trick to realize that if the displacement between the rabbit and fox is $\mathbf{r}(t) = (x(t), y(t))$, then the quantity $r + x$ is conserved. To see this, let θ be the angle between the rabbit and fox's velocity vectors. Then

$$\frac{dr}{dt} = -v + v\cos\theta$$

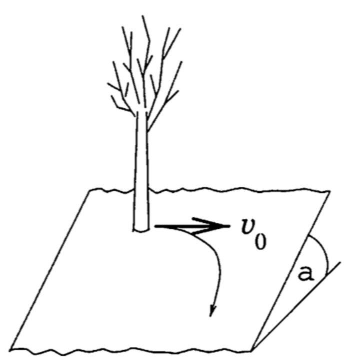
because of the fox's chasing and rabbit's motion, and

$$\frac{dx}{dt} = v - v\cos\theta$$

because of the rabbit's motion and fox's chasing. Then $r + x$ is constant. Initially $r + x = a + 0 = a$, and after a long time $r = x = d$, so the final separation is $d = a/2$.

[2] **Problem 19.** Suppose the fox in the above example instead has speed $u > v$. How long does it take to catch the rabbit?

[2] **Problem 20** (PPP 85)**.** A child is on an icy hill, which may be modeled as an inclined plane.



The coefficient of friction $\mu_k = \mu_s$ is small enough so that, if the child gets the tiniest push, she will begin sliding down the plane. Now suppose the child gets a horizontal push, with initial speed v_0.

What is the child's final speed?

[4] **Problem 21.** EuPhO 2023, problem 2. (Warning: the algebra will be a bit messy.)

3 Motion in Two Dimensions

Idea 7

Often, motion in two dimensions can be treated as two independent one-dimensional problems. A change of reference frame may be necessary first.

Idea 8

In problems involving an inclined plane, it helps to draw the inclination angle θ to be less than about 30°. This reduces mistakes, because almost every angle will be either θ or $90^\circ - \theta$, and you can identify which by sight.

Example 5

Consider projectile motion where wind provides a constant horizontal force F. At what angle should a projectile of mass m be launched in order to return to the thrower?

Solution

The key idea is to use tilted coordinate systems. Clearly, when the only force is downward, the projectile must be launched straight upward. Now, the horizontal force acts like an effective horizontal gravitational acceleration of F/m, so that gravity is effectively tilted an angle $\tan^{-1}(F/mg)$ away from the vertical. One must launch the projectile directly "upward" with respect to this effective gravitational field, so the launch angle is an angle $\tan^{-1}(F/mg)$ from the vertical. (For a related problem, see the infamous $F = ma$ 2014 problem 19.)

Example 6: $F = ma$ 2022 A23

For projectiles, the force of air resistance can be modeled as proportional to the speed ("linear drag") or proportional to the square of the speed ("quadratic drag"), depending on the circumstances. Two identical objects, A and B, are dropped from the same height h simultaneously, but object A is given an initial horizontal velocity v. The objects hit the ground at times t_A and t_B. How do these times compare, assuming linear or quadratic drag?

Solution

For linear drag, the horizontal and vertical components of the motion are independent,

$$a_x = -bv_x, \quad a_y = -g - bv_y$$

for some coefficient b. That means the time to hit the ground, which depends on the vertical motion, is independent of the initial horizontal velocity, so $t_A = t_B$. But for quadratic drag,

$$a_y = -g - bv_y|v|$$

which means the upward drag force is larger when the horizontal velocity is larger, so $t_A > t_B$.

Since the components are independent for linear drag, it's not too hard to write down an expression for the trajectory, by recycling the results of example 1. But for quadratic drag, the results of problem 2 won't help much; the two-dimensional problem is much harder.

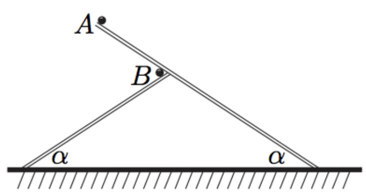
[1] **Problem 22** (Quarterfinal 2002)**.** A cart is rigged with a vertical cannon so that, when the cart is stationary on a horizontal track, the cannonball is fired straight up and lands back in the cannon. In each of the following situations, does the cannonball land back in the cannon, in front of it, or behind it?

(a) The cart is moving on a frictionless horizontal track with speed v.

(b) The cart is accelerating down a frictionless inclined track with angle θ.

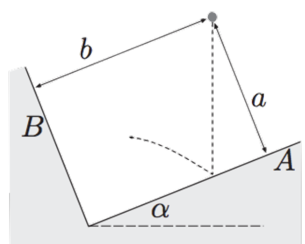
(c) The cart is accelerating down an inclined track with angle θ, and friction slows it down.

[2] **Problem 23** (Kalda)**.** Two balls at points A and B are released from rest at the same moment, from the locations shown below. All surfaces are frictionless.



If it takes time t_A and t_B for the balls to hit the ground, at what time was the distance between the balls the smallest?

[2] **Problem 24** (Kalda)**.** Two planar frictionless walls are placed at right angles, where wall A makes an angle α to the horizontal. A perfectly elastic ball is released from rest at a point a distance a from wall A and b from wall B.



After a long time, what is the ratio of the number of times the ball has bounced against wall B to the number of times it has bounced against wall A?

[2] **Problem 25.** USAPhO 2004, problem A4.

[3] **Problem 26** (EFPhO 2010)**.** A sprinkler can be modeled as a small hemisphere on the ground. Water shoots out from the hemisphere in all directions, with speed v perpendicular to the hemisphere.

(a) Find the total surface area of ground watered by the sprinkler.

(b) At what distance from the sprinkler does the ground get the wettest?

[3] **Problem 27.** USAPhO 2023, problem A1. A neat exercise on collisions and projectile motion.

Example 7

A bug flies towards a light with constant speed v, always making an angle α with the radial direction. If the initial distance to the lamp is L and the radius of the lamp is R, through what total angle does it turn before hitting the lamp?

Solution

In this case we can't avoid solving differential equations, but they're not too hard. It's easiest to work in polar coordinates, with the center of the lamp at the origin. By decomposing the velocity into radial and tangential components, we have

$$\frac{dr}{dt} = -v\cos\alpha, \quad r\frac{d\theta}{dt} = v\sin\alpha.$$

We only care about the path, not the time-dependence, so we divide these equations to get

$$\frac{dr}{d\theta} = -\frac{r}{\tan\alpha}$$

where we manipulated differentials as in **P1**. Separating and integrating,

$$-\int_L^R \frac{dr}{r} = \frac{\Delta\theta}{\tan\alpha}$$

which tells us that

$$\Delta\theta = (\tan\alpha)\log\frac{L}{R}.$$

The shape traced out is a logarithmic spiral.

[2] **Problem 28.** The pilot of a supersonic jet airplane wishes to make a big noise at the origin by flying around it in a path such that all of the noise he makes is heard simultaneously at the origin. The jet travels with Mach number $M > 1$, meaning that its speed is M times the speed of sound. If the pilot starts at $(r, \theta) = (a, 0)$, find the pilot's path $r(\theta)$.

[4] **Problem 29.** Consider a mass m on a table attached to a spring at the origin with zero relaxed length, which exerts the force

$$\mathbf{F} = -k\mathbf{r}$$

on the mass. We will find the general solution for $\mathbf{r}(t) = (x(t), y(t))$ in two different ways.

(a) Directly write down the answer, using the fact that the x and y coordinates are independent.

(b) Sketch a representative sample of solutions. What kind of curve does the trajectory follow?

(c) $\star$ Here's a more unusual way to arrive at the same answer. Go to a noninertial reference frame rotating with angular velocity ω_0 about the origin, so that the centrifugal force cancels out the spring force. In this frame, the only relevant force is the Coriolis force $-2m\boldsymbol{\omega}_0 \times \mathbf{v}$. Find the general solution in this frame, then transform back to the original frame and show that you get the same answer as in part (a). (This can get a bit messy; the easiest way is to treat the plane as the complex plane, i.e. work in terms of the variable $r = x + iy$.)

4 Optimal Launching

Finally, we'll consider projectile motion questions that involve optimization. These are rare on the USAPhO, but they are quite fun problems, with occasionally very slick solutions.

Example 8

A bug wishes to jump over a cylindrical log of radius R lying on the ground, so that it just grazes the top of the log horizontally as it passes by. What is the minimum launch speed v required to do this?

Solution

Let P be the point at the top of the log. For the bug to be moving horizontally at P, energy conservation applied to the vertical motion gives an initial v_y obeying

$$\frac{1}{2}mv_y^2 = 2mgR, \quad v_y = 2\sqrt{gR}.$$

Thus, we need to find the minimum v_x for the motion to be possible. If v_x is too low, the hypothetical trajectory of the bug will instead pass through the log. At the lowest possible v_x, the bug's trajectory is not just tangent to the log at point P, but also has the same radius of curvature (i.e. the trajectory and the log's shape have the same first and second derivatives).

For uniform motion in a circle of radius r, the acceleration is $a = v^2/r$. Conversely, when an object follows a trajectory of instantaneous radius of curvature r, its acceleration component normal to the path must be $a = v^2/r$. So applying this to the bug at P gives

$$g = \frac{v_x^2}{R}, \quad v_x = \sqrt{gR}.$$

Thus, the minimum initial speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5}\, gR.$$

This radius of curvature trick doesn't come up often, but it's cool when it does.

[2] **Problem 30.** NBPhO 2020, problem 3. A nice warmup for the problems below.

[3] **Problem 31.** An object is launched from the top of a hill, where the ground lies an angle ϕ below the horizontal. Show that the range of a projectile is maximized if it is launched along the angle bisector of the vertical and the ground.

[3] **Problem 32** (PPP 35)**.** A point P is located above an inclined plane with angle α. It is possible to reach the plane by sliding under gravity down a straight frictionless wire, joining P to some point P' on the plane. Geometrically, how should P' be chosen so as to minimize the time taken? (Hint: think about the set of points that can be reached for all possible angles of the wire, after time t.)

Idea 9

Since mechanics is time-reversible, and the speed of a projectile only depends on its height and not the path taken, finding the way to reach point B from point A with the lowest possible initial speed is the same as finding the way to reach point A from point B with the lowest possible initial speed.

[4] **Problem 33.** Two fences of heights h_1 and h_2 are erected on a horizontal plain, so that the *tops* of the fences are separated by a distance d. Show that the minimum speed needed to throw a projectile over both fences is $\sqrt{g(h_1 + h_2 + d)}$.

[4] **Problem 34.** Problems 31 and 33 can be solved with pure geometry. Consider the set of points, in two dimensions, that a projectile can reach with a fixed initial speed v and a fixed launch point. It turns out that the boundary of this set (i.e. the curve of points that a projectile can just barely reach) is a vertical parabola with its focus at the launch point. A parabola is defined as the set of points whose distance to the focus equals the distance to a line, called the directrix.

(a) Show that trajectories that touch the parabola must be tangent to it.

(b) Show that if a point is hit with the smallest possible initial speed, then the initial velocity must be perpendicular to the final velocity.

(c) Using the geometric definition of a parabola, recover the answers to problems 31 and 33.

If you really like this kind of thing, you can try Physics Cup 2019, problem 3, with solutions here.

[3] **Problem 35.** IPhO 2012, problem 1A.

5 Reading Graphs

In some kinematics problems, you'll have to infer what's going on from a diagram. To make progress, you'll have to print out the diagram to make measurements directly on it.

[3] **Problem 36.** EFPhO 2015, problem 6.

[3] **Problem 37.** EFPhO 2008, problem 3.

Remark

For a harder problem from the same genre, see EuPhO 2019, problem 3. Almost all competitors received zero points on it, largely because it relies on a specialized trick introduced earlier in this problem set. You can try it for entertainment if you have time and really like kinematics. The official solutions are here.