Mechanics IV: Oscillations

Chapter 4 of Morin covers oscillations, as does chapter 10 of Kleppner and Kolenkow, and chapter 10 of Wang and Ricardo, volume 1. For a deeper treatment that covers normal modes in more detail, see chapters 1 through 6 of French. Jaan Kalda also has short articles on using Lagrangian-like techniques and the adiabatic theorem. For some fun discussion, see chapters I-21 through I-25, II-19, and II-38 of the Feynman lectures. There is a total of 88 points.

1 Small Oscillations

Idea 1

If an object obeys a linear force law, then its motion is simple harmonic. To compute the frequency, one must the restoring force per unit displacement. More generally, if the force an object experiences can be expanded in a Taylor series with a nonzero linear restoring term, the motion is approximately simple harmonic for small displacements. (However, don't forget that there are also situations where oscillations are not even approximately simple harmonic, no matter how small the displacements are.)

Example 1: KK 4.13

The Lennard–Jones potential

$$U(r) = \epsilon \left(\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right)$$

is commonly used to describe the interaction between two atoms. Find the equilibrium radius and the angular frequency of small oscillations about this point for two identical atoms of mass m bound to each other by the Lennard–Jones interaction.

Solution

To keep the notation simple, we'll set $\epsilon = r_0 = 1$ and restore them later. The equilibrium radius is the radius where the derivative of the potential vanishes, and

$$U'(r) = -12r^{-13} + 12r^{-7} = 0$$

implies that that the equilibrium radius is $r = r_0$. Because the force accelerates both of the atoms, the angular frequency is

$$\omega = \sqrt{\frac{U''(r)}{m/2}}$$

where $m/2$ is the so-called reduced mass. At the equilibrium point, we have

$$U''(r_0) = (12)(13)r_0^{-14} - (12)(7)r_0^{-8} = 72.$$

Restoring the dimensionful factors, we have $U''(r_0) = 72\epsilon/r_0^2$, so

$$\omega = \frac{12}{r_0}\sqrt{\frac{\epsilon}{m}}.$$

[3] **Problem 1** (Morin 5.13)**.** A hole of radius R is cut out from an infinite flat sheet with mass per unit area σ. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

(a) What is the force on a mass m that is located on L, a distance x from the center of the hole? (Hint: consider the plane to consist of many concentric rings.)

(b) Now suppose the particle is released from rest at this position. If $x \ll R$, find the approximate angular frequency of the subsequent oscillations.

(c) Now suppose that $x \gg R$ instead. Find the period of the resulting oscillations.

(d) Now suppose the mass begins at rest on the plane, but slightly displaced from the center. Do oscillations occur? If so, what is the approximate frequency?

[2] **Problem 2.** Some questions about small oscillations with the buoyant force.

(a) A cubical glacier of side length L has density ρ_i and floats in water with density ρ_w. Find the angular frequency of small oscillations, assuming that a face of the glacier always remains parallel to the water surface, and that the force of the water on the glacier is always given by the hydrostatic buoyant force.

(b) A ball of radius R floats in water with half its volume submerged. Find the angular frequency of small oscillations, making the same assumption.

(c) There are important effects that both of the previous parts neglect. What are some of them? Is the true oscillation frequency higher or lower than the one found here?

[3] **Problem 3.** 🕔 USAPhO 1998, problem A2. To avoid some confusion, skip part (a), since there actually isn't a nice closed-form expression for it.

[3] **Problem 4.** 🕔 USAPhO 2009, problem A3.

[3] **Problem 5.** 🕘 USAPhO 2010, problem B1.

Example 2

Find the acceleration of an Atwood's machine with masses m and M and a massless pulley and string.

Solution

The standard way to do this is to let a_1 and a_2 be the accelerations of the masses, let T be the unknown tension in the string, solve for T by setting a_1 and a_2 to have equal magnitudes, then plug T back in to find the common acceleration. This procedusre is unnecessarily complicated, because we are using two coordinates when the string really ensures the system has only a single degree of freedom.

We can alternatively use energy conservation in terms of "generalized coordinates". Let q be a coordinate which describes "how much the string has moved along itself". In other words,

$q = 0$ initially, and when $q = q_0$, the mass M has moved down by q_0 and the mass m has moved up by q_0. The kinetic and potential energies of the system are simply

$$K = \frac{1}{2}(m+M)\dot{q}^2, \quad V = qg(m-M).$$

To find the acceleration $\ddot{q}$, we differentiate energy conservation with respect to time,

$$0 = \frac{d(K+V)}{dt} = (m+M)\ddot{q}\dot{q} + \dot{q}g(m-M).$$

Solving gives the familiar result

$$\ddot{q} = \frac{M-m}{M+m}\,g.$$

Intuitively, we could say that from the standpoint of this generalized coordinate, the "total force" is $(M-m)g$, and the "total inertia" is $M+m$.

Idea 2

The idea shown in example 2 is very general. Consider *any* system whose configuration can be described by a single "generalized coordinate" q. If its energy can be decomposed into a kinetic energy that depends only on $\dot{q}$ and a potential energy that depends only on q,

$$K = K(\dot{q}), \quad V = V(q)$$

then the energy conservation equation $d(K+V)/dt = 0$ can be used to find the generalized acceleration $\ddot{q}$. Explicitly, the chain rule tells us that

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} = -\frac{\partial V}{\partial q}.$$

As a simple example, for a particle moving in one dimension, taking $q = x$ gives

$$\frac{d}{dt}(m\dot{x}) = -\frac{dV}{dx}$$

which is the usual Newton's second law. So in general, we will call $\partial K/\partial \dot{q}$ a "generalized momentum", and $-\partial V/\partial q$ a "generalized force". Note that static equilibrium can occur when $\partial V/\partial q = 0$, which is just the principle of virtual work from **M2**.

Remark

The result above is a special case of the Euler–Lagrange equation in Lagrangian mechanics, which states that if a system is described by a Lagrangian L, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

When $L = K(\dot{q}) - V(q)$, we recover the previous result. But more generally, it might not be possible to meaningfully decompose L into a "kinetic" and "potential" piece at all! We won't use this more general form below. While it is more powerful, it is also more complicated, and if you find yourself using it for an Olympiad problem, there's probably an easier way.

[1] **Problem 6.** A rope is nestled inside a curved frictionless tube. The rope has a total length ℓ and uniform mass per length λ. The shape of the tube can be arbitrarily complicated, but the left end of the rope is higher than the right end by a height h. If the rope is released from rest, find its acceleration. (For a related question, see $F = ma$ 2019 B24.)

Idea 3

Generalized coordinates are really useful for problems that involve complicated objects but only have one relevant degree of freedom, which is especially true for oscillations problems. For instance, if the kinetic and potential energy have the form

$$K = \frac{1}{2} m_{\text{eff}} \dot{q}^2, \quad V = \frac{1}{2} k_{\text{eff}} q^2$$

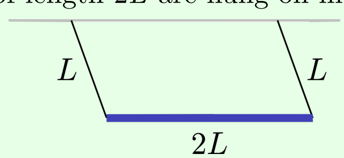
then the oscillation's angular frequency is always

$$\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}.$$

Note that q need not have units of position, m_{eff} need not have units of mass, and so on. When $V(q)$ is a more general function, we can expand it about a minimum q_{min}, so that $k_{\text{eff}} = V''(q_{\text{min}})$. This lets us avoid dealing with possibly complicated constraint forces.

Example 3: $F = ma$ 2022A #10

The two ends of a uniform rod of length $2L$ are hung on massless strings of length L.



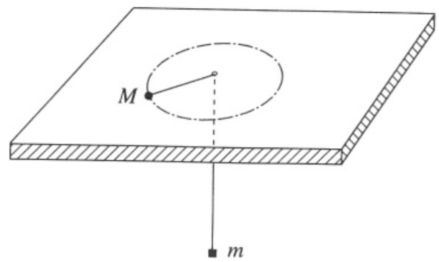
If the strings are attached to the ceiling, and the rod is pulled a small distance horizontally and released as shown, what is the period of oscillation?

Solution

This kind of question becomes completely trivial when you use the above idea. Using the rotation angle θ as a generalized coordinate, K and V are both exactly the same as for a simple pendulum of length L, because the rod doesn't rotate, so the period is $2\pi\sqrt{L/g}$.

[3] **Problem 7.** A particle in a uniform vertical gravitational field is constrained to move on a curve $y(x)$. If $y(x)$ is a circular arc, then this system is just a simple pendulum, and we know that its period is not perfectly independent of its amplitude. Find a differential equation relating dy/dx and y, so that the period of oscillation is *exactly* $2\pi/\omega_0$ for a fixed parameter ω_0, independent of amplitude. (Hint: work in terms of the coordinate s, the arc length along the curve.)

[3] **Problem 8** (Cahn). A particle of mass M is constrained to move on a frictionless horizontal plane. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane.



The system is set up so that the mass M moves in a circle of radius r, while the mass m remains still. Show that this motion is stable with respect to small changes in r, and find the angular frequency of small oscillations.

[4] **Problem 9.** IPhO 1984, problem 2. If you use the energy methods above, you won't actually need to know anything about fluid mechanics to do this nice, short problem!

2 Springs and Pendulums

Now we'll consider more general problems involving springs and pendulums, two very common components in mechanics questions. As a first example, we'll use the fictitious forces met in **M2**.

Example 4: PPP 79

A pendulum of length ℓ and mass m initially hangs straight downward in a train. The train begins to move with uniform acceleration a. If a is small, what is the period of small oscillations? If a can be large, is it possible for the pendulum to loop over its pivot?

Solution

The fictitious force in the train's frame due to the acceleration is equivalent to an additional, horizontal gravitational field, so the effective gravity is

$$\mathbf{g}_{\text{eff}} = -a\hat{\mathbf{x}} - g\hat{\mathbf{y}}.$$

For small oscillations, we know the period is $2\pi\sqrt{L/g}$ in ordinary circumstances. By precisely the same logic, it must be replaced with

$$T = 2\pi\sqrt{\frac{L}{g_{\text{eff}}}} = 2\pi\frac{\sqrt{L}}{(g^2+a^2)^{1/4}}.$$

As a gets larger, the effective gravity points closer to the horizontal. In the limit $g/a \to 0$, the effective gravity is just horizontal, so the pendulum oscillates about the horizontal. Its endpoints are the downward and upward directions, so it never can get past the pivot.

Here's a follow-up question: if the train can decelerate quickly, how should you stop it so that the pendulum doesn't end up swinging at the end? The most efficient way is to first quickly decelerate to half speed, which, in the frame of the train, provides a horizontal impulse to the pendulum. Then wait a half-period $\pi\sqrt{L/g}$, so that the pendulum's momentum turns around, and then quickly stop, providing a second impulse that precisely cancels the pendulum's horizontal motion. Tricks like this are used by crane operators to transport loads, and by physicists to transport clouds of ultracold atoms without warming them up.

Example 5

If a spring with spring constant k_1 and relaxed length ℓ_1 is combined with a spring with spring constant k_2 and relaxed length ℓ_2, find the spring constant and relaxed length of the combined spring, if the combination is in series or in parallel.

Solution

For the series combination, the new relaxed length is clearly $\ell = \ell_1 + \ell_2$. Suppose the first spring is stretched by x_1 and the second by x_2. The tensions in the springs must balance,

$$F = k_1 x_1 = k_2 x_2.$$

Thus, the new spring constant is

$$k = \frac{F}{x_1 + x_2} = \frac{k_2 x_2}{x_2(k_2/k_1 + 1)} = \frac{k_1 k_2}{k_1 + k_2}.$$

For example, if the spring is cut in half, the pieces have spring constant $2k$.

Now consider the parallel combination. In this case it's clear that the new spring constant is $k = k_1 + k_2$, since the tensions of the springs add. The new relaxed length ℓ is when the forces in the springs cancel out, so

$$k_1(\ell - \ell_1) + k_2(\ell - \ell_2) = 0$$

which implies

$$\ell = \frac{k_1 \ell_1 + k_2 \ell_2}{k_1 + k_2}.$$

[2] **Problem 10** (Morin 4.20)**.** A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

[3] **Problem 11** (Morin 4.22)**.** A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal.

(a) Geometrically, what kind of curve is the resulting trajectory?

(b) Find the value of v so that the projectile hits the ground traveling straight downward.

[5] **Problem 12.** A uniform spring of spring constant k and total mass m is attached to the wall, and the other end is attached to a mass M.

(a) Show that when $m \ll M$, the oscillation's angular frequency is approximately

$$\omega = \sqrt{\frac{k}{M + m/3}}.$$

(b) **[A]** ⋆ Generalize part (a) to arbitrary values of m/M. (Hint: to begin, approximate the massive spring as a finite combination of smaller massless springs and point masses, as in the example in **M2**. It will not be possible to solve for ω in closed form, but you can get a compact implicit expression for it. Check that it reduces to the result of part (a) for small m/M, and interpret the results for large m/M. This is a challenging problem that requires almost all the techniques we've seen so far; you might want to return to it after doing section 4.)

[2] **Problem 13** (PPP 77)**.** A small bob of mass m is attached to two light, unstretched, identical springs. The springs are anchored at their far ends and arranged along a straight line. If the bob is displaced in a direction perpendicular to the line of the springs by a small length ℓ, the period of oscillation of the bob is T. Find the period if the bob is displaced by length 2ℓ.

[3] **Problem 14.** USAPhO 2015, problem A3.

[3] **Problem 15.** USAPhO 2008, problem B1.

Example 6

About how accurately can you measure g with a simple pendulum?

Solution

This simple question illustrates how rich experimental physics can be, even in elementary settings. First, let's think about the uncertainties in the pendulum's length and period.

- Length: a reasonable length for an experiment is $L \sim 1\,\text{m}$. We should use a wire, not a string, to avoid stretching. If you measure the wire with a good ruler, you can get down to $\Delta L \sim 1\,\text{mm}$. If you use calipers, you can get $\Delta L \sim 0.1\,\text{mm}$. Assuming the latter gives a fractional uncertainty $\Delta L/L \sim 10^{-4}$.

- Period: if the length is a meter, the period will be $T \simeq 2\,\text{s}$. (This isn't a total coincidence! 17$^{\text{th}}$ century scientists proposed to define the standard unit of length precisely so this would be true.) One might estimate the timing uncertainty to be given by human reaction speed, $\Delta T \sim 250\,\text{ms}$, but this is too pessimistic, because you can see the pendulum coming. An extensive study of manual timing at swimming competitions found a typical spread $\Delta T \sim 70\,\text{ms}$. Moreover, since a pendulum's motion is regular, you can "lock in" with your sense of rhythm to do even better than this. Finally, we can let the pendulum swing for $N = 100$ consecutive periods and measure the total time. These improvements allow a timing uncertainty $\Delta T/(NT) \sim 10^{-4}$.

Combining these results with the error propagation rules of **P2**, we can estimate $\Delta g/g \sim 10^{-4}$ for a well-performed experiment. But any real experiment also has to contend with systematic effects which can bias the results. Let's consider and estimate a couple of them.

- The bob has finite size, so the pendulum is really a physical pendulum. We can estimate the size of this effect by thinking about how much the bob's size changes the pendulum's moment of inertia. If the bob has radius $r \sim 1\,\text{cm}$, the change is roughly $r^2/L^2 \sim 10^{-4}$.

- The wire isn't massless, so the effective length of the pendulum is less than L. If we use a lead bob whose mass is a few kilograms, and the wire is a thin steel wire whose mass is a few grams, the effect is roughly $m_{\text{wire}}/m_{\text{bob}} \sim 10^{-3}$.

- The motion has finite amplitude θ_0. As we saw in **P1**, this changes the period fractionally by $\theta_0^2/16$, and for an amplitude of a few degrees this is $\sim 10^{-3}$.
- The pendulum oscillates in air. This leads to two distinct effects: the buoyant force on the bob decreases the effective value of g, and the "virtual mass" effect, discussed further in **M7**, increases the bob's effective inertia. These effects shift the period in the same direction, and they are both of order $\rho_{\text{air}}/\rho_{\text{bob}} \sim (1\,\text{kg/m}^3)/(10^4\,\text{kg/m}^3) \sim 10^{-4}$.
- The Earth is rotating, leading to centrifugal and Coriolis forces. The latter turns out to be unimportant; as shown in **M6**, it rotates the pendulum's plane of oscillation, rather than shifting its period. Unless you're conducting the experiment in Greenland or Antarctica, the centrifugal force produces a shift of order $\omega_E^2 R_E/g \sim 10^{-3}$.
- The pendulum's motion is slightly damped, which lengthens the oscillation period. This factor depends on how frictionless the support is. However, if it was set up so that 100 consecutive periods can be measured, one must have quality factor $Q \gtrsim 10^3$. One can show that the fractional shift in frequency is $\sim 1/Q^2 \sim 10^{-6}$.

There are plenty of other factors, but these are the most important ones, and a few of them are larger than the uncertainty from the length and period. But the good thing is that all of them can be calculated, and thereby subtracted out, leading to an ultimate final precision of $\Delta g/g \sim 10^{-4}$. That is indeed the best precision achieved during the 1800s, through extensive effort. For real measurements and further details, see this paper.

3 Damped and Driven Oscillations

We now review damped oscillators, which we saw in **M1**, and consider driven oscillators. For more guidance, see sections 4.3 and 4.4 of Morin.

[2] **Problem 16.** Consider a damped harmonic oscillator, which experiences force $F = -bv - kx$.

(a) As in **M1**, show that the general solution for $x(t)$ is

$$x(t) = A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

and solve for the $\omega_\pm$.

(b) For sufficiently small b, the roots are complex. In this limit, show that by taking the real part, one finds an exponentially damped sinusoidal oscillation. Roughly how many oscillation cycles happen when the amplitude damps by a factor of e?

(c) For large b, the roots are pure imaginary, the position simply decays exponentially, and we say the system is overdamped. Find the condition for the system to be overdamped.

[4] **Problem 17.** Analyzing a damped and driven harmonic oscillator.

(a) Consider a damped harmonic oscillator which experiences a driving force $F = F_0 \cos(\omega t)$. Passing to complex variables, Newton's second law is

$$m\ddot{x} + b\dot{x} + kx = F_0 e^{i\omega t}.$$

If $x(t)$ is a complex exponential, then we know that the left-hand side is still a complex exponential, with the same frequency. This motivates us to guess $x(t) = A_0 e^{i\omega t}$. Show that this solves the equation for some A_0.

(b) Of course, the general solution needs to be described by two free parameters, to match the initial position and velocity. Argue that it takes the form

$$x(t) = A_0 e^{i\omega t} + A_+ e^{i\omega_+ t} + A_- e^{-i\omega_- t}$$

where the $\omega_\pm$ are the ones you found in problem 16.

(c) After a long time, the "transient" $A_\pm$ terms will decay away, leaving the steady state solution

$$x(t) \approx A_0 e^{i\omega t}$$

which oscillates at the same frequency as the driving. The actual position is the real part,

$$x(t) \approx |A_0| \cos(\omega t - \phi)$$

where $A_0 = |A_0| e^{-i\phi}$. Evaluate $|A_0|$ and ϕ.

(d) Sketch the amplitude $|A_0|$ and phase shift ϕ as a function of ω. Can you intuitively see they take the values they do, for ω small, $\omega \approx \sqrt{k/m}$, and ω large?

(e) There are several distinct things people mean when they speak of "resonant frequencies". Find the driving angular frequency ω that maximizes (i) the amplitude $|A_0|$, (ii) the amplitude of the velocity, and (iii) the average power absorbed from the driving force. (As you'll see, these are all about the same when the damping is weak, so the distinction between these isn't so important in practice.)

[3] **Problem 18.** The quality factor of a damped oscillator is defined as $Q = m\omega_0/b$, where $\omega_0 = \sqrt{k/m}$. It measures both how weak the damping is, and how sharp the resonance is.

(a) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{total energy of the oscillator}}{\text{average energy dissipated per radian}}.$$

Then estimate Q for a guitar string.

(b) Show that for a lightly damped oscillator,

$$Q \approx \frac{\text{resonant frequency}}{\text{width of resonance curve}}$$

where the width of the resonance curve is defined to be the range of driving frequencies for which the amplitude is at least $1/\sqrt{2}$ the maximum.

For more about Q, see pages 424 through 428 of Kleppner and Kolenkow.

The next two problems explore other ways of driving harmonic oscillators.

[2] **Problem 19.** Consider a pendulum which can perform small-angle oscillations in a plane with natural frequency f. The pendulum bob is attached to a string, and you hold the other end of the string in your hand. There are three simple ways to drive the pendulum:

(a) Move the end of the string horizontally with sinusoidal frequency f'.

(b) Move the end of the string vertically with sinusoidal frequency f'.

(c) Apply a quick rightward impulse to the bob with frequency f'.

In each case, for what value(s) of f' can the amplitude become large? (This question should be done purely conceptually; don't write any equations, just visualize it!)

[5] **Problem 20.** GPhO 2016, problem 1. Record your answers on the official answer sheet.

4 Normal Modes

Idea 4: Normal Modes

A system with N degrees of freedom has N normal modes when displaced from equilibrium. In a normal mode, the positions of the particles are of the form $x_i(t) = A_i \cos(\omega t + \phi_i)$. That is, all particles oscillate with the same frequency. Normal modes can be either guessed physically, or found using linear algebra as explained in section 4.5 of Morin.

The general motion of the system is a superposition of these normal modes. So to compute the time evolution of the system, it's useful to decompose the initial conditions into normal modes, because they all evolve independently by linearity.

Example 7

Two blocks of mass m are connected with a spring of spring constant k and relaxed length L. Initially, the blocks are at rest at positions $x_1(0) = 0$ and $x_2(0) = L$. At time $t = 0$, the block on the right is hit, giving it a velocity v_0. Find $x_1(t)$ and $x_2(t)$.

Solution

The equations of motion are

$$m\ddot{x}_1 = k(x_2 - x_1 - L)$$
$$m\ddot{x}_2 = k(x_1 + L - x_2).$$

The system must have two normal modes. The obvious one is when the two masses oscillate oppositely, $x_1 = -x_2$. The other one is when the two masses move parallel to each other, $x_1 = x_2$, and this normal mode formally has zero frequency. The initial condition is the superposition of these two modes.

We can show this a bit more formally. Define the normal mode amplitudes u and v as

$$x_1 = \frac{u-v}{2}, \quad x_2 = \frac{u+v}{2}.$$

Solving for u and v, we find

$$u = x_1 + x_2, \quad v = x_2 - x_1.$$

Using the equations of motion for x_1 and x_2, we have the equations of motion

$$\ddot{u} = 0, \quad m\ddot{v} = -2k(v - L)$$

which just verifies that the normal modes are independent, with angular frequency zero and $\omega = \sqrt{2k/m}$ respectively. We can fit the initial condition if

$$u(0) = L, \quad v(0) = L, \quad \dot{u}(0) = v_0, \quad \dot{v}(0) = v_0.$$

The normal mode amplitudes are then

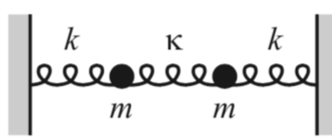
$$u(t) = L + v_0 t, \quad v(t) = L + \frac{v_0}{\omega}\sin\omega t.$$

Plugging this back in gives

$$x_1(t) = \frac{v_0 t}{2} - \frac{v_0}{2\omega}\sin\omega t, \quad x_2(t) = L + \frac{v_0 t}{2} + \frac{v_0}{2\omega}\sin\omega t.$$

Each mass is momentarily stationary at time intervals of $2\pi/\omega$, though neither mass ever moves backwards. If you didn't know about normal modes, you could also arrive at this conclusion by playing around with the equations; you could see that they decouple when you add and subtract them, for instance.

[3] **Problem 21** (Morin 4.10)**.** Three springs and two equal masses lie between two walls, as shown.

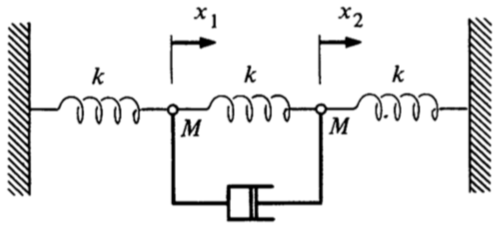


The spring constant k of the two outside springs is much larger than the spring constant $\kappa \ll k$ of the middle spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. If the initial positions are given by $x_1(0) = a$ and $x_2(0) = 0$, and if both masses are released from rest, show that

$$x_1(t) \approx a\cos((\omega+\epsilon)t)\cos(\epsilon t), \quad x_2(t) \approx a\sin((\omega+\epsilon)t)\sin(\epsilon t)$$

where $\omega = \sqrt{k/m}$ and $\epsilon = (\kappa/2k)\omega$. Explain qualitatively what the motion looks like. This is an example of beats, which result from superposition two oscillations of nearly equal frequencies; we will see more about them in **W1**.

[3] **Problem 22** (KK 10.11)**.** Two identical particles are hung between three identical springs.



Neglect gravity. The masses are connected as shown to a dashpot which exerts a force bv, where v is the relative velocity of its two ends, which opposes the motion.

(a) Find the equations of motion for x_1 and x_2.

(b) Show that the equations of motion can be solved in terms of the variables $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$.

(c) Show that if the masses are initially at rest and mass 1 is given an initial velocity v_0, the motion of the masses after a sufficiently long time is

$$x_1(t) = x_2(t) = \frac{v_0}{2\omega} \sin \omega t$$

and evaluate ω.

Example 8

Three identical masses are connected by three identical springs, forming an equilateral triangle in equilibrium. Describe the normal modes of the system.

Solution

Let the system be confined to the xy plane. Then there are three masses that each can move in two dimensions, giving six degrees of freedom. Since we must be able to construct the general solution by superposing normal modes, there should be six normal modes. They are:

- Uniform translation. This yields two independent normal modes, as you can superpose motion in any two distinct directions (e.g. along the x and y axes) to get motion in any direction. These modes have zero frequency, since $\sin(\omega t) \propto t$ in the limit $\omega \to 0$.
- Uniform rotation about the axis of symmetry.
- A "breathing" motion where the whole triangle expands and contracts.
- A "scissoring" motion where one mass moves outward and the other two move inward. You might think there are three scissoring normal modes, but they are redundant: just like how the three sides of the equilateral triangle lie in a plane, these three normal modes formally lie in a plane, in the sense that you can superpose any two of them to get the third. So there are two independent scissoring modes.

Thus we have six normal modes, as expected. If the system can move in three-dimension space, we need three more; they are uniform translation in the z direction, and rotation about the x and y axes.

[5] **Problem 23** (Morin 4.12, IPhO 1986). N identical masses m are constrained to move on a horizontal circular hoop connected by N identical springs with spring constant k. The setup for $N = 3$ is shown below.

(a) Find the normal modes and their angular frequencies for $N = 2$.

(b) Do the same for $N = 3$.

(c) ⋆ Do the same for general N. (Hint: the normal modes you found in part (a) should have each mass oscillating with unit amplitude, but a different phase. Try to write the normal modes in part (b) in the same form, and then guess a pattern.)

(d) If one of the masses is replaced with a mass $m' \ll m$, qualitatively describe how the set of frequencies changes.

(e) Now suppose the masses alternate between m and $m' \ll m$. Qualitatively describe the set of frequencies.

Part (c) will be useful in **X1**, where we will quantize the normal modes found here.

[4] **Problem 24.** [A] In this problem, you will analyze the normal modes of the double pendulum, which consists of a pendulum of length ℓ and mass m attached to the bottom of another pendulum, of length ℓ and mass m. To solve this problem directly, one has to compute the tension forces in the two strings, which are quite complicated. A much easier method is to use energy.

(a) Parametrize the position of the pendulum in terms of the angle θ_1 the top string makes with the vertical, and the angle θ_2 the bottom string makes with the vertical. Write out the kinetic energy K and the potential energy V to second order in the θ_i and $\dot{\theta}_i$.

(b) The Euler–Lagrange equations for the system are

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\theta}_i} = -\frac{\partial V}{\partial \theta_i}.$$

Using the results of part (a), write these equations in the form

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\frac{g}{L}\, A \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

where A is a 2×2 matrix. This is a generalization of $\ddot{\theta} = -g\theta/L$ for a single pendulum.

(c) Find the normal modes and their angular frequencies, using the general method in section 4.5 of Morin.

Remark

We mostly considered examples with two or three masses, but the techniques above work for systems with arbitrarily many degrees of freedom. However, this quickly becomes intractable unless the setup is highly symmetric, as in problem 23. Without such symmetry, a computer is generally necessary, so this sort of question won't appear on standard Olympiads. However, if you're curious, see ITPO 2016, problem 1 and Physics Cup 2021, problem 3 for examples.

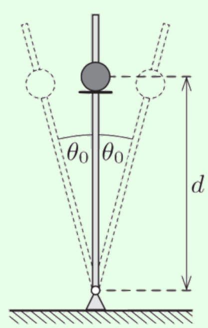
5 [A] Adiabatic Change

Idea 5

When a problem contains two widely separate timescales, such as a fast oscillation superposed on a slow overall motion, one can solve for the fast motion while neglecting the slow motion, then solve for the slow motion by replacing the fast motion with an appropriate average.

Example 9: MPPP 21

A small smooth pearl is threaded onto a rigid, smooth, vertical rod, which is pivoted at its base. Initially, the pearl rests on a small circular disc that is concentric with the rod, and attached to it a distance d from the rotational axis. The rod starts executing simple harmonic motion around its original position with small angular amplitude θ_0.



What angular frequency of oscillation is required for the pearl to leave the rod?

Solution

The reason the pearl leaves the rod is that the normal force rapidly varies in direction, with an average upward component. If this average upward force is greater than gravity, the pearl accelerates upward and leaves the rod.

In this case, the fast motion is the oscillation of the rod, while the slow motion is the rate of change of the pearl's distance from the pivot, which can be neglected during one oscillation. The pearl has horizontal displacement and acceleration

$$x(t) = -d \sin\theta \approx -d\theta(t) = -\theta_0 d \sin\omega t, \quad a_x(t) = \theta\omega^2 d \sin\omega t.$$

This is supplied by the horizontal component of the normal force. The vertical component is

$$N_y = N_x \tan\theta(t) \approx ma_x(t)\theta(t) = m\theta_0^2\omega^2 d\sin^2\omega t.$$

Now we average over the fast motion to understand the slow motion. Since the average value of $\sin^2(\omega t)$ is $1/2$, the condition for the pearl to go up is

$$\frac{1}{2}m\theta_0^2\omega^2 d > mg$$

which gives

$$\omega > \frac{1}{\theta_0}\sqrt{\frac{2g}{d}}.$$

Example 10

A mass m oscillates on a spring with spring constant k_0 with amplitude A_0. Over a very long period of time, the spring smoothly and continuously weakens until its spring constant is $k_0/2$. Find the new amplitude of oscillation.

Solution

In this case the fast motion is the oscillation of the mass, while the slow motion is the weakening of the spring. We can solve the problem by considering how the energy changes in each oscillation, due to the slight decrease in k.

Suppose that the spring constant drops in one instant by a factor of $1-\epsilon$. Then the kinetic energy stays the same, while the potential energy drops by a factor of $1-\epsilon$. Since the kinetic and potential energy are equal on average, this means that if the spring constant gradually decreases by a factor of $1-x$ over a full cycle, with $x \ll 1$, then the energy decreases by a factor of $1-x/2$.

The process finishes after N oscillations, where $(1-x)^N \approx e^{-Nx} = 1/2$. At this point, the energy has dropped by a factor of $(1-x/2)^N \approx e^{-Nx/2} = 1/\sqrt{2}$. Since the energy is also $kA^2/2$, the new amplitude is $\sqrt[4]{2}A_0$.

Amazingly, the question can also be solved in one step using a subtle conserved quantity.

Solution

Sinusoidal motion is just a projection of circular motion. In particular, it's equivalent to think of the mass as being tied to a spring of zero rest length attached to the origin, and performing a circular orbit about the origin, with the "actual" oscillation being the x component. (This is special to zero-length springs obeying Hooke's law, and occurs because the spring force $-k\mathbf{x} = -k(x, y)$ has its x-component independent of y, and vice versa.)

Since the spring constant is changed gradually, the orbit has to remain circular. Then *angular* *momentum* is conserved, and we have

$$L \propto vr = \omega A^2 \propto \sqrt{k} A^2.$$

Then the final amplitude is $\sqrt[4]{2} A_0$ as before.

Both of these approaches are tricky. The energy argument is very easy to get wrong, while the angular momentum argument seems to come out of nowhere and is inapplicable to other situations. But the formal angular momentum here turns out to be a special case of a more general conserved quantity, which is useful in a wide range of similar problems.

Idea 6: Adiabatic Theorem

If a particle performs a periodic motion in one dimension in a potential that changes very slowly, then the "adiabatic invariant"

$$I = \oint p \, dx$$

is conserved. This integral is the area of the orbit in phase space, an abstract space whose axes are position and momentum.

Solution

Since the potential changes slowly, the energy is roughly conserved in each oscillation cycle,

$$E = \frac{p^2}{2m} + \frac{1}{2} k x^2.$$

Thus, within one oscillation cycle, the curve $p(x)$ traces out an approximate ellipse in phase space, with semimajor and semiminor axes of $\sqrt{2mE}$ and $\sqrt{2E/k}$. Over the course of many oscillations, the energy changes, but the area of this ellipse is the adiabatic invariant,

$$I = \oint p \, dx = \pi \sqrt{2mE} \sqrt{2E/k} = 2\pi E \sqrt{\frac{m}{k}} \propto A^2 \sqrt{km}.$$

Thus, $A \propto k^{-1/4}$ in an adiabatic change of k, recovering the answer found earlier.

Remark

The existence of the adiabatic invariant is hard to see in pure Newtonian mechanics, but it falls naturally out of Hamiltonian mechanics, which is built on phase space. In fact, Hamiltonian mechanics makes a lot of useful facts easier to see, which is why it's the most commonly used foundation for introducing quantum mechanics. It is commonly introduced at the end of an undergraduate upper-division mechanics course, and therefore beyond the Olympiad syllabus. If you'd like to learn more about Hamiltonian mechanics, or just see how the adiabatic theorem is derived, see David Tong's lecture notes.

The conservation of the adiabatic invariant has important consequences throughout physics.

As we'll see in problem 25 and in **X1**, it ensures that the conditions which determine energy levels in quantum mechanics remain true as a system is changed. As we'll discuss in **R3**, the adiabatic invariant is also useful to analyze the motion of charges in magnetic fields.

It's also closely connected to adiabatic processes in thermodynamics. You've probably heard that an adiabatic thermodynamic process has to be *fast*, so that no heat exchange can happen. But the more fundamental definition is that it's *slow*, relative to the dynamics of the particles. In this case, the conservation of the adiabatic invariant for each particle implies the conservation of the entropy of the gas. That's because, as we'll discuss in **T2**, the entropy fundamentally measures the volume of phase space that the system can occupy.

[3] **Problem 25.** Consider a pendulum whose length adiabatically changes from L to $L/2$.

(a) If the initial (small) amplitude was θ_0, find the final amplitude using the adiabatic theorem.

(b) Give a physical interpretation of the adiabatic invariant.

(c) When quantum mechanics was being invented, it was proposed that the energy in a pendulum's oscillation was always a multiple of $\hbar\omega$, where ω is the angular frequency. At the first Solvay conference of 1911, Lorentz asked whether this condition would be preserved upon slow changes in the length of the pendulum, and Einstein said yes. Reproduce Einstein's analysis.

[4] **Problem 26.** A block of mass M and velocity v_0 to the right approaches a stationary puck of mass $m \ll M$. There is a wall a distance L to the right of the puck.

(a) Assuming all collisions are elastic, find the minimum distance between the block and the wall by explicitly analyzing each collision. (Note that it does not suffice to just use the adiabatic theorem, because it applies to slow change, while the collisions are sharp. Nonetheless, you should find a quantity that is approximately conserved after many collisions have occurred.)

(b) Approximately how many collisions occur before the block reaches this minimum distance?

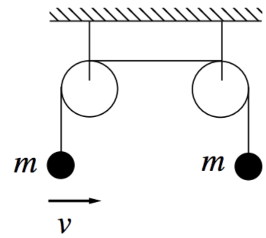
(c) The adiabatic index γ is defined so that PV^γ is conserved during an adiabatic process. In one dimension, the volume V is simply the length, and P is the average force. Using the adiabatic theorem, infer the value of γ for a one-dimensional monatomic gas.

[3] **Problem 27.** A block of mass m is bouncing back and forth in a box spanning $0 < x < L$, with initial speed v_0. At time $t = 0$, the potential energy is slowly raised in part of the box, so that

$$V(x,t) = \begin{cases} V_0 & 0 < x < ut \\ 0 & ut < x < L \end{cases}$$

where $V_0 > mv_0^2/2$, and the speed of the potential u is very small. At time $t = L/u$, when the potential covers the entire box, what is the block's speed?

[4] **Problem 28 ($\boldsymbol{F = ma}$, BAUPC).** Two particles of mass m are connected by pulleys as shown.



The mass on the left is given a small horizontal velocity v, and oscillates back and forth.

(a) Without doing any calculation, which mass is higher after a long time?

(b) Verify your answer is right by computing the average tension in the leftward string, in the case where the other end of the string is fixed, for amplitude $\theta_0 \ll 1$.

(c) Let the masses begin a distance L from the pulleys. Find the speed of the mass which eventually hits the pulley, at the moment it does, in terms of L and the initial amplitude θ_0.