Mechanics V: 2D Rotation

Two-dimensional rotation is covered in chapter 6 of Kleppner, chapter 8 of Morin, or chapters 3 and 5 of Wang and Ricardo, volume 1. Further discussion is given in chapters I-18 and I-19 of the Feynman lectures. For more on dot and cross products, see the first two lectures of MIT OCW 18.02. There is a total of **86** points.

1 2D Rotational Kinematics

Idea 1

It can be shown that the instantaneous velocity of a two-dimensional rigid body can always be written as pure rotation about some point $\mathbf{r}_0$, not necessarily in the body. Then the velocity of a point in the body at location $\mathbf{r}$ is

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0).$$

This equation defines $\boldsymbol{\omega}$, the angular velocity vector, which points out of the page. Differentiating gives the acceleration of a point in the body,

$$\begin{aligned}\mathbf{a} &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_0) \\ &= \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_0) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)) - \boldsymbol{\omega} \times \mathbf{v}_0\end{aligned}$$

where $\mathbf{v}_0 = d\mathbf{r}_0/dt$ is the rate of change of the location of the pivot point. (Note that this differs from the velocity of the point in the body instantaneously at the pivot, which is always zero.) However, this latter expression is often hard to use, because you usually won't know $\mathbf{r}_0(t)$, or it'll have a complicated form.

Alternatively, we can write the velocity in terms of translation plus pure rotation about any desired point, which is almost always chosen to be the center of mass. This gives

$$\mathbf{v} = \mathbf{v}_{\mathrm{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}).$$

The $\boldsymbol{\omega}$ here is the same as in the previous expression. Differentiating gives the acceleration,

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_{\mathrm{CM}}) \\ &= \mathbf{a}_{\mathrm{CM}} + \boldsymbol{\alpha} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\mathrm{CM}})).\end{aligned}$$

If you need an acceleration, this form tends to be easiest to use. The three terms represent the acceleration of the center of mass, the angular acceleration, and the centripetal acceleration, written in a slightly fancy way.

Remark 1: Cross Products

The cross product of two vectors is antisymmetric and distributes over addition,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \quad \mathbf{a} \times (r_1 \mathbf{b}_1 + r_2 \mathbf{b}_2) = r_1 \mathbf{a} \times \mathbf{b}_1 + r_2 \mathbf{a} \times \mathbf{b}_2.$$

The latter means a cross product can be differentiated using the product rule. Moreover,

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}.$$

These are all you need to compute any cross product, but it's also helpful to get geometric intuition. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then the direction of $\mathbf{c}$ is found by applying the right-hand rule to $\mathbf{a}$ and $\mathbf{b}$, and its magnitude is $|\mathbf{a}||\mathbf{b}| \sin\theta$ where θ is the angle between them. Finally, when differentiating a cross product, the ordinary product rule applies (just like for dot products).

Example 1

Describe the velocities of points in a disc rolling without slipping using both methods.

Solution

Let the disc have radius R, lie in the xy plane, and roll along the x axis. Consider the moment where the bottom of the disc touches the origin. At this moment its motion can be thought of as pure rotation about the origin,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -\omega\hat{\mathbf{z}} \times \mathbf{r}.$$

On the other hand, the motion can also be thought of as simultaneous translation of the center of mass and rotation about the center of mass, so

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_{\text{CM}} + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_{\text{CM}}) \\ &= \omega R\hat{\mathbf{x}} - \omega\hat{\mathbf{z}} \times (\mathbf{r} - R\hat{\mathbf{y}}) \\ &= \omega R\hat{\mathbf{x}} - \omega\hat{\mathbf{z}} \times \mathbf{r} - \omega R\hat{\mathbf{x}} \\ &= -\omega\hat{\mathbf{z}} \times \mathbf{r}. \end{aligned}$$

As we can see, both decompositions are completely equivalent. Which one you want to use depends on the problem; you might even use both in the same problem.

Remark

As we'll discuss in **M8**, for three-dimensional rigid bodies, rotation is always about an *axis*. When people say a body rotates about a *point*, as we will throughout this problem set, they always mean that the body is two-dimensional and moves in a fixed plane, and the "point" of rotation is the intersection of the 3D rotation axis with the plane.

You might also wonder how the description

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

applies in the case where a body is in pure translation. That's simply the case where $\boldsymbol{\omega}$ goes to zero while $\mathbf{r}_0$ gets infinitely far away. In this limit, $\mathbf{r}$ is negligible, so every point in the body moves with the same velocity $\mathbf{v} = -\boldsymbol{\omega} \times \mathbf{r}_0$. So the description still works for pure translation, though it's a bit unnatural.

Example 2

A cylinder of circumference 10 cm is placed on a table. You place a ruler horizontally on the cylinder, so that initially the top of the cylinder touches the 0 cm mark on the ruler. By pushing the ruler forward, you make the cylinder complete one full revolution, without anything slipping. At this point, what mark on the ruler is touching the top of the cylinder?

Solution

In one revolution, the center of the cylinder moves forward by 10 cm. The top of the cylinder moves forward by 20 cm, so the ruler also moves forward by this distance. The relative motion of the ruler and cylinder is the difference, which is 10 cm.

[4] **Problem 1.** Some brief puzzles about rotation.

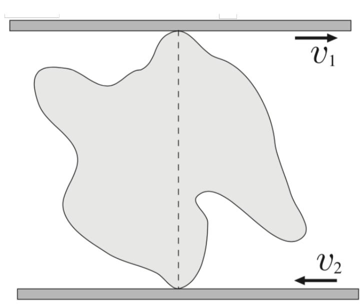
(a) Consider two identical coins laid flat on a table. One is fixed in place, and the second is rolled without slipping around the first. Once the second coin's center has returned to its original position, how many times has it rotated? (Be sure to check your answer experimentally!)

(b) A bicycle wheel is rolling without slipping. When it is photographed, its spokes look blurred, except along a curve of special points, which don't look blurred at all. What is this curve?

(c) Consider a spaceship floating in space, without any thrusters that can expel material. Conservation of momentum implies that it cannot move its center of mass. But is it possible to turn the spaceship around? In other words, is it possible for it to begin stationary in one orientation, and end up stationary in another orientation? If so, why doesn't this violate conservation of angular momentum?

(d) Hold out your arm with your elbow bent at 90° and your palm straight out, facing down. Find a way to end up in the same position but with your palm facing up, without ever bending or rotating your wrist.

[2] **Problem 2** (Kalda)**.** A rigid lump is squeezed between two places, one of which is moving at velocity v_1 and the other at v_2. At some moment, the velocities are horizontal and the two contact points are vertically aligned.



Indicate geometrically all of the points in the body with speed either v_1 or v_2.

[2] **Problem 3.** USAPhO 2010, problem A1.

2 Moments of Inertia

Idea 2

For a two-dimensional object, the moment of inertia

$$I = \int x^2 + y^2 \, dm$$

about the origin obeys the parallel axis theorem

$$I = I_{\text{CM}} + M r_{\text{CM}}^2$$

where I_{CM} is the moment of inertia about the center of mass, and M is the total mass. Defining I_x and I_y to be the moment of inertia about the x and y, we have

$$I = I_x + I_y, \quad I_x = \int y^2 \, dm, \quad I_y = \int x^2 \, dm$$

which is called the perpendicular axis theorem.

[3] **Problem 4.** Basic moment of inertia computations.

(a) Compute the moment of inertia for an $L_x \times L_y$ rectangular plate about an axis passing perpendicular to it through the center.

(b) Compute the moment of inertia for a uniform disc of radius R and mass M, about an axis perpendicular to it through its center. What about an axis lying in the disc, passing through its center?

(c) Compute the moment of inertia of a uniform solid cone of mass M, with height H and a base of radius R, about its symmetry axis.

[2] **Problem 5 ($\boldsymbol{F = ma}$ 2016 24).** The moment of inertia of a uniform equilateral triangle with mass m and side length a about an axis through one of its sides and parallel to that side is $ma^2/8$. What is the moment of inertia of a uniform regular hexagon of mass m and side length a about an axis through two opposite vertices?

3 Rotational Dynamics

In this section we'll consider some dynamic problems involving rotation.

Idea 3: Angular Momentum

For a system of particles we define the angular momentum and torque

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i, \quad \boldsymbol{\tau} = \sum_i \mathbf{r}_i \times \mathbf{F}_i, \quad \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

Using the first part of idea 1, we may write the angular momentum of a rigid body as

$$\mathbf{L} = I\boldsymbol{\omega}, \quad K = \frac{1}{2} I \omega^2$$

where I is the moment of inertia about the instantaneous axis of rotation. Alternatively, using the second part,

$$\mathbf{L} = I_{\text{CM}}\boldsymbol{\omega} + \mathbf{r}_{\text{CM}} \times M\mathbf{v}_{\text{CM}}, \quad K = \frac{1}{2} I_{\text{CM}}\omega^2 + \frac{1}{2} M v_{\text{CM}}^2$$

where M is the total mass. Both forms are useful in different situations. Systems cannot exert torques on themselves, provided they obey the strong form of Newton's third law: the force between two objects is equal and opposite, and directed along the line joining them.

Idea 4

The idea above refers to taking torques about a fixed point, but often it is easier to consider a moving point P. Let $\mathbf{L}$ be the angular momentum about point P in the frame of P, i.e. the frame whose axes don't rotate, but whose origin follows P around. Working in this frame will produce fictitious forces, since P can accelerate. Such forces act at the center of mass, just like gravity.

The upshot is that if P is the center of mass, then the fictitious force in the frame of P will produce no "fictitious torque". So it's safe to use $\boldsymbol{\tau} = d\mathbf{L}/dt$ about either a fixed point, or in the frame of the center of mass.

Idea 5

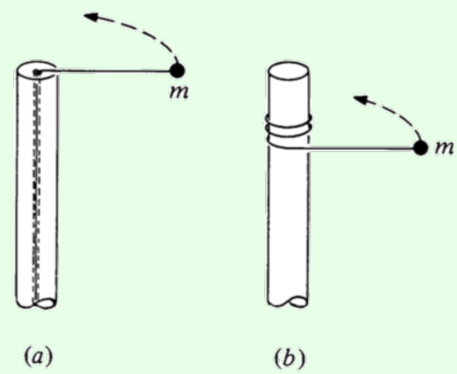
There is a third, more confusing way of applying $\boldsymbol{\tau} = d\mathbf{L}/dt$ that you might rarely see: taking torques about the instantaneous center of rotation. In general, this doesn't work, because the instantaneous center of rotation can accelerate, producing an extra fictitious torque as mentioned above.

However, it turns out this procedure gives the correct answer if the object is instantaneously at rest. That's why taking torques about the contact point for the spool in **M2** to find the initial angular acceleration was valid. It wouldn't have been valid at any instant afterward, after the spool had picked up some velocity.

For more discussion of this subtlety, which isn't mentioned in any textbooks I know of, see the paper *Moments to be cautious of*.

Example 3: KK 6.13

A mass m is attached to a post of radius R by a string. Initially it is a distance r from the center of the post and is moving tangentially with speed v_0. In case (a) the string passes through a hole in the center of the post at the top. The string is gradually shortened by drawing it through the hole. In case (b) the string wraps around the outside of the post. Ignore gravity.



For each case, find the final speed of the mass when it hits the post.

Solution

In case (a), the energy isn't conserved, since work is done on the mass as it moves inward. (Physically, we can see this by noting there could be a weight slowly descending on the other end of the string.) However, angular momentum conservation says $Rv = rv_0$, so $v = rv_0/R$.

If you don't believe in angular momentum conservation yet, it's not too hard to show this with $F = ma$ as well. Let the tangential and radial speeds of the mass be v_t and v_r, where $v_r \ll v_t$. Since v_r is nonzero, there is a component of acceleration parallel to the velocity,

$$\frac{T}{m}\sin\theta \approx \frac{v_t^2}{r}\frac{v_r}{v_t}$$

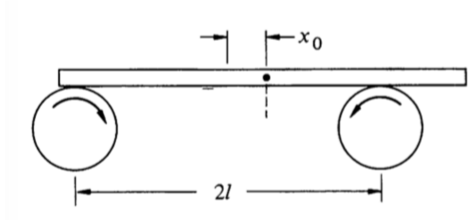
and this is equal to the rate of change of speed, which to first order in v_r/v_t is dv_t/dt. Thus,

$$\frac{dv_t}{dt} = \frac{v_r v_t}{r} = -\frac{v_t}{r}\frac{dr}{dt}$$

from which we conclude rv_t is constant, as expected. (As mentioned in **M2**, you never *need* ideas like torque and angular momentum. Life is just harder without them.)

In case (b), the angular momentum about the axis of the pole isn't conserved, since the tension force has a lever arm about that axis. However, the mass's energy is conserved. A simple physical way to see this is to note that the massless string can't store any energy, and the post doesn't do work on the string, which means the string can't do any work on the mass. Thus, the final speed is just $v = v_0$. (Of course, if you don't believe in energy conservation, you could get the same result by showing that the trajectory of the mass is always perpendicular to the string, though this takes more work.)

[2] **Problem 6** (KK 6.9). A heavy uniform bar of mass M rests on top of two identical rollers which are continuously turned rapidly in opposite directions, as shown.

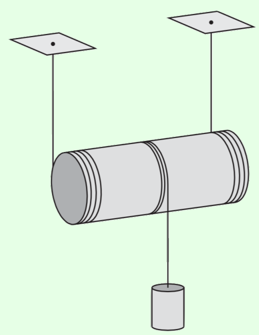


The centers of the rollers are a distance 2ℓ apart. The coefficient of friction between the bar and the roller surfaces is μ, a constant independent of the relative speed of the two surfaces. Initially the bar is held at rest with its center at distance x_0 from the midpoint of the rollers. At time $t = 0$ it is released. Find the subsequent motion of the bar.

[2] **Problem 7** (BAUPC). A mass is connected to one end of a massless string, the other end of which is connected to a very thin frictionless vertical pole. The string is initially wound completely around the pole, in a very large number of small horizontal circles, with the mass touching the pole. The mass is released, and the string gradually unwinds. What angle does the string make with the pole when it becomes completely unwound? (Though the setup is similar to that of example 3, you can't ignore gravity here.)

Example 4: MPPP 49

A uniform rod of mass M and radius R is attached to two identical strings. The strings are wound around the cylinder as shown, and their free ends are fastened to the ceiling.



A third cord is attached to and wound around the middle of the cylinder, and a mass M is attached to the other side. There is sufficient friction so that the strings do not slip. Find the acceleration of the mass immediately after release.

Solution

Let a be the downward acceleration of the center of mass of the rod, let T_1 be the total tension in the first two strings, and let T_2 be the tension in the third. The rod rolls without slipping about its contact axis with the first two strings, which means the downward acceleration of the mass is $a_{\text{mass}} = 2a$.

The Newton's second law equations are thus

$$Ma = T_2 + Mg - T_1, \quad 2Ma = Mg - T_2$$

for the rod and mass. Taking torques about the axis of the rod gives

$$(T_1 + T_2)R = \frac{1}{2}MR^2\,\alpha$$

and using $a = \alpha R$ converts this to

$$Ma = 2T_1 + 2T_2.$$

We now have three equations in three unknowns, so we can straightforwardly solve to find $a = (6/11)g$. This implies that the acceleration of the mass is

$$a_{\text{mass}} = \frac{12}{11}g.$$

Done, right? No, this is the wrong answer! Since the acceleration is greater than free fall, the tension T_2 must be negative. But a string can't support a negative tension, so it instead goes slack. The mass thus free falls, so $a_{\text{mass}} = g$.

In retrospect, we could have seen this conclusion with less work. Suppose the mass were not attached. Then the acceleration of the rod can be computed with the standard rolling without slipping formula,

$$a = \frac{g\sin\theta}{1+\beta} = \frac{g}{1+\beta}, \quad I = \beta MR^2.$$

For *any* (axially symmetric) mass distribution in the rod, we have $0 \leq \beta \leq 1$. The acceleration of the part where the mass would have been attached is hence

$$a_{\text{mass}} = \frac{2g}{1+\beta} \geq g.$$

This implies that any string we attach there must go slack immediately after release.

Example 5

If you're riding a bike and need to stop quickly, what are the advantages and disadvantages of using the front brake versus the rear brake?

Solution

Work in the reference frame moving with the bike. In this frame, the backward friction force is balanced by a forward friction force on the center of mass; the combination of the two produces a torque that tends to lift the rear wheel off the ground. If you use the front brake, you can stop more quickly, because the normal force on the front tire stays higher. But if you brake too hard with the front brake, you could flip yourself over the handlebars. This can't happen when using the rear brake alone, because the brake stops doing anything the moment the rear wheel lifts off the ground.

Idea 6

It is often useful in rotational dynamics to treat the rotational and linear motion of a rigid body conceptually separately.

Example 6: $F = ma$ 2018 B23

Two particles with mass m_1 and m_2 are connected by a massless rigid rod of length L and placed on a horizontal frictionless table. At time $t = 0$, the first mass receives an impulse perpendicular to the rod, giving it speed v. At this moment, the second mass is at rest. When is the next time the second mass is at rest?

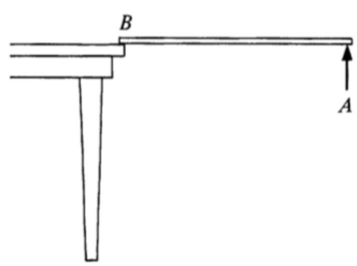
Solution

The motion is the superposition of two motions: uniform translation of both masses with speed $m_1 v/(m_1 + m_2)$ and circular motion about the common center of mass, where the two masses have speeds $m_2 v/(m_1 + m_2)$ and $m_1 v/(m_1 + m_2)$, respectively. This ensures that the second mass begins at rest and the first mass has speed v.

The circular part of the motion determines when the second mass will be at rest again. The radius of the circle the second mass makes is its distance from the center of mass, $Lm_1/(m_1 + m_2)$. This gives a period of

$$t = \frac{2\pi L m_1/(m_1 + m_2)}{m_1 v/(m_1 + m_2)} = \frac{2\pi L}{v}.$$

[2] **Problem 8** (KK 6.14)**.** A uniform stick of mass M and length ℓ is suspended horizontally with end B on the edge of a table, while end A is held by hand.

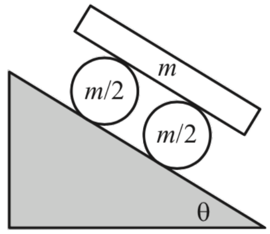


Point A is suddenly released. Right after release, find the vertical force at B, as well as the downward acceleration of point A. You should find a result greater than g. Explain how this can be possible, given that gravity is the only downward external force in the problem.

[2] **Problem 9.** Quarterfinal 2005, problem 4. This is a neat example of separating out rotational and translational motion. For a similar idea, see Morin 8.73.

[2] **Problem 10** (Morin 8.71)**.** A ball sits at rest on a piece of paper on a table. You pull the paper in a straight line out from underneath the ball. You are free to pull the paper in an arbitrary way forward or backwards; you may even jerk it so that the ball starts to slip. After the ball comes off the paper, it will eventually roll without slipping. Show that, in fact, the ball ends up at rest. Is it possible to pull the paper in such a way that the ball ends up exactly where it started?

[2] **Problem 11** (Morin 8.28)**.** Consider the following "car" on an inclined plane.



The system is released from rest, and there is no slipping between any surfaces. Find the acceleration of the board.

[2] **Problem 12.** USAPhO 2006, problem A1.

[2] **Problem 13.** USAPhO 2013, problem A2.

[3] **Problem 14.** USAPhO 2014, problem A1.

[3] **Problem 15.** A uniform stick of length L and mass M begins at rest. A massless rocket is attached to the end of the stick, and provides a constant force F perpendicular to the stick. Find the speed of the center of mass of the stick after a long time. Ignore gravity. You may find the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

which we first encountered in **P1**, useful.

[4] **Problem 16** (KK 6.41)**.** A plank of length $2L$ leans nearly vertically against a wall. All surfaces are frictionless. The plank starts to slip downward. Find the height of the top of the plank when it loses contact with the wall or floor.

Example 7: EFPhO 2013

A uniform ball and a uniform ring are both released from rest from the same height on an inclined plane with inclination angle θ. They arrive at the bottom of the plane in time T_B and T_R, respectively. The coefficients of friction of both objects with the plane are $\mu_k = 0.3$ and $\mu_s = 0.5$. Find the ratio T_B/T_R as a function of the angle θ.

Solution

When rolling without slipping, the acceleration of an object with moment of inertia $\beta m R^2$ about its center of mass is

$$a = \frac{g \sin\theta}{1+\beta}$$

as mentioned in a previous example. The tangential force from friction is thus

$$f = mg\sin\theta\,\frac{\beta}{1+\beta}$$

which means rolling without slipping occurs when

$$\mu_s mg\cos\theta \geq mg\sin\theta\frac{\beta}{1+\beta}$$

or equivalently

$$\tan\theta \leq \mu_s \frac{1+\beta}{\beta}.$$

For the ball, this is when $\theta \leq 60.3^\circ$, and for the ring $\theta \leq 45^\circ$. Whenever either object slips, its acceleration is instead $a = g(\sin\theta - \mu_k \cos\theta)$.

Since the motion is uniformly accelerated, $T_B/T_R = \sqrt{a_R/a_B}$. For $\theta \leq 45^\circ$, both roll without slipping, so the formula above applies, giving a ratio of

$$\frac{T_B}{T_R} = \sqrt{\frac{1+\beta_B}{1+\beta_R}} = \sqrt{\frac{7}{10}}.$$

For $\theta \geq 60.3^\circ$ they both slip, so the ratio is unity. For the angles in between, the ring slips, giving a slightly more complicated expression. At the boundaries between these three regimes, the ratio T_B/T_R jumps discontinuously.

The next two problems require careful thought, and test your understanding of the multiple ways to describe rotational kinematics and dynamics. It will be useful to review idea 1.

[3] **Problem 17.** USAPhO 1999, problem B1.

[3] **Problem 18.** USAPhO 2019, problem B3. It's worth reading the solution carefully afterward.

4 Rotational Collisions

Idea 7: Angular Impulse

During a collision with impulse $\mathbf{J}$, the angular momentum changes by the "angular impulse" $\mathbf{r} \times \mathbf{J}$. In many problems involving collisions which conserve angular momentum, energy is necessarily lost in the collision process. This is another example of an inherently inelastic process, an idea we first encountered in **M3**.

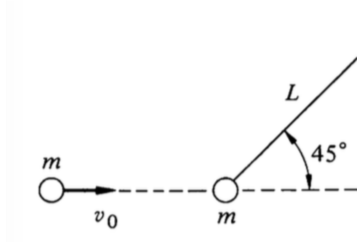
[3] **Problem 19** (Morin 8.22)**.** A uniform ball of radius R and mass m rolls without slipping with speed v_0. It encounters a step of height h and rolls up over it.

(a) Assuming that the ball sticks to the step during this process, show that for the ball to climb over the step,

$$v_0 \geq \sqrt{\frac{10gh}{7}} \left(1 - \frac{5h}{7R}\right)^{-1}.$$

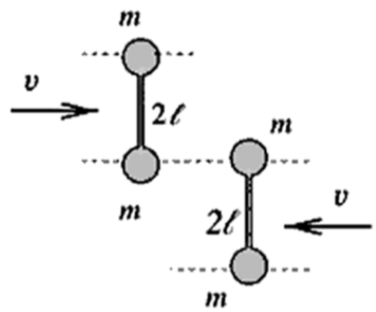
(b) Energy is lost to heat by the inelastic collision of the ball with the step. In the limit of small h, how much heat is produced?

[3] **Problem 20** (KK 6.38)**.** A rigid massless rod of length L joins two particles, each of mass m. The rod lies on a frictionless table, and is struck by a particle of mass m and velocity v_0 as shown.



After an elastic collision, the projectile moves straight back. Find the angular velocity of the rod about its center of mass after the collision.

[3] **Problem 21** (PPP 47)**.** Two identical dumbbells move towards each other on a frictionless table as shown.



Each consists of two point masses m joined by a massless rod of length 2ℓ. The dumbbells collide elastically; describe what happens afterward.

[3] **Problem 22.** USAPhO 2014, problem B1.

[3] **Problem 23.** EuPhO 2024, problem 1. A nice exercise on the process of a rotational collision.

[4] **Problem 24.** EuPhO 2018, problem 1. An elegant rotation problem.

5 Rotational Oscillations

In this section we'll consider small oscillations problems involving rotation.

Idea 8

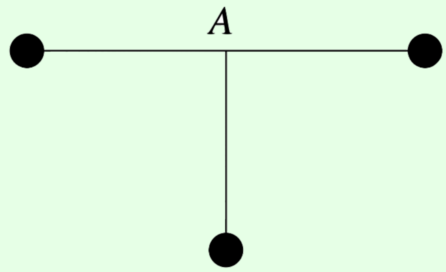
A physical pendulum is a rigid body of mass m pivoted a distance d from its center of mass, with moment of inertia I about the pivot. When considering physical pendulums, we always assume the pivot exerts no torque on the pendulum; that is, it is a "simple support", providing no bending moment, as discussed in **M2**. This is a good approximation if the pivot is smooth and small. In this case, the angular frequency for small oscillations is

$$\omega = \sqrt{\frac{mgd}{I}}.$$

For some neat real-world applications of this formula, see this paper.

Example 8: $F = ma$ 2018 A14

Three identical masses are connected with identical rigid rods and pivoted at point A.



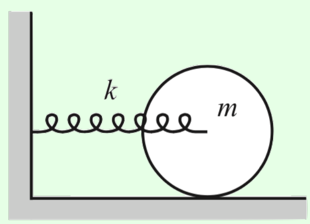
If the lowest mass receives a small horizontal push to the left, it oscillates with period T_1. If it receives a small push into the page, it oscillates with period T_2. Find the ratio T_1/T_2.

Solution

Both modes are physical pendulums, which have period proportional to $\sqrt{I/Mgx}$ where x is the distance from the pivot to the center of mass, and I is the moment of inertia about the pivot. Since x is the same in both cases, $T_1/T_2 \propto \sqrt{I_1/I_2} = \sqrt{3}$, because in the second case only the bottom mass contributes to the moment of inertia.

Example 9: Morin 8.41

The axis of a solid cylinder of mass m and radius r is connected to a spring of spring constant k, as shown.



If the cylinder rolls without slipping, find the angular frequency of the oscillations.

Solution

This is a question best handled using the energy methods of **M4**. The potential energy is $kx^2/2$ as usual, where x describes the position of the cylinder's center of mass. The kinetic energy is $mv^2/2 + I\omega^2/2 = (3/4)mv^2$, since the cylinder is rolling without slipping. Therefore

$$\omega = \sqrt{\frac{k}{m_{\text{eff}}}} = \sqrt{\frac{2k}{3m}}.$$

More complicated variants of this kind of problem can be solved in a similar way.

Example 10: Russia 2011

A uniform ring of mass m and radius r is suspended symmetrically on three inextensible strings of length ℓ. Find the angular frequency of small oscillations.

Solution

The small oscillations are torsional, i.e. the ring rotates about its axis of symmetry. When the ring has twisted by an angle θ, the strings are an angle $\phi \approx (r/\ell)\theta$ from the vertical. Thus, summing over the three strings, the restoring torque is

$$\tau \approx -mgr\phi \approx -\frac{mgr^2}{\ell}\,\theta.$$

Setting this equal to $I\alpha$, we find $\omega = \sqrt{g/\ell}$.

The tricky thing about this problem is that it's harder to solve with the energy method. If you try, you immediately run into the problem that there seems to be no potential energy anywhere, since the strings don't stretch! The source of the potential energy is that the ring moves up a small amount as it oscillates, since the strings are no longer vertical,

$$h = \ell - \sqrt{\ell^2 - r^2\theta^2} \approx \frac{r^2\theta^2}{2\ell}.$$

Therefore we have

$$K = \frac{1}{2}mr^2\dot{\theta}^2, \quad V = \frac{1}{2}\frac{mgr^2}{\ell}\theta^2$$

and the answer follows as usual. (There is also a kinetic energy contribution from the ring's vertical motion, but it's negligible.) The lesson here is that the force/torque and energy approach have different strengths. The energy approach is often easier because it lets you ignore some internal details of the system. But it can be harder because it requires you to understand the kinematics of the system to second order, rather than first order.

[2] **Problem 25.** A circular pendulum consists of a point mass m on a string of length ℓ, which is made to rotate in a horizontal circle. By using only the equation $\boldsymbol{\tau} = d\mathbf{L}/dt$ about an origin of your choice, compute the angular frequency if the string makes a constant angle θ with the horizontal.

[3] **Problem 26.** Using a physical pendulum, one can measure the acceleration due to gravity as

$$g = \frac{4\pi^2}{T^2}\frac{I}{md}.$$

In practice, I is not very precisely known, since it depends on the exact shape of the material. Kater found an ingenious way to circumvent this problem. We pivot the pendulum at an arbitrary point and measure the period T. Next, by trial and error, we find another pivot point which has the same period, which lies at a different distance from the center of mass.

(a) Show that

$$g = \frac{4\pi^2 L}{T^2}$$

where L is the sum of the lengths from these points to the CM. This allows a measurement of g without knowledge of the moment of inertia about the center of mass.

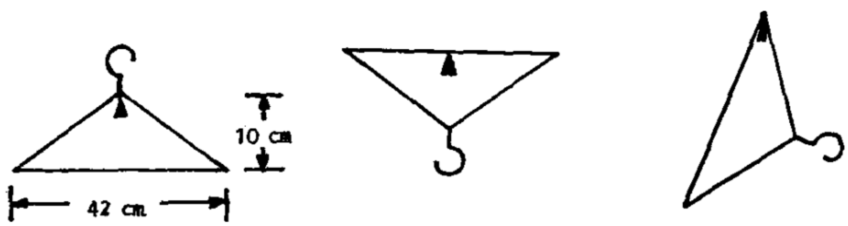
(b) When the two pivot points lie on a line, on opposite sides of the center of mass, then L is simply the distance between the pivot points, which can be measured quite precisely, removing the need to find the center of mass. However, we do have to be a bit careful. Show that the formula in part (a) gives a totally wrong answer for a uniform rod of length L, with pivot points at its two ends. What's going on, and how can we fix the problem?

[3] **Problem 27.** USAPhO 1999, problem A4.

[3] **Problem 28.** USAPhO 2011, problem B2.

[3] **Problem 29.** USAPhO 2002, problem B1. This one is trickier than it looks! It can be solved with either a torque or energy analysis, but both require care.

[4] **Problem 30** (IPhO 1982)**.** A coat hanger can perform small oscillations in the plane of the figure about the three equilibrium figures shown.



In the first two, the long side is horizontal. The other two sides have equal length. The period of oscillation is the same in all cases. The coat hanger does *not* necessarily have uniform density. Where is the center of mass, and how long is the period?

[4] **Problem 31** (APhO 2007)**.** A uniform ball of mass M and radius r is encased in a thin spherical shell, also of mass M. The shell is placed inside a fixed spherical bowl of radius R, and performs small oscillations about the bottom. Assume that friction between the bowl and shell is very large, so the shell essentially always rolls without slipping.

The ball is made of an unusual material: it can quickly transition between a liquid and solid state. When the ball is in the liquid state, it has no viscosity, and hence no friction with the shell. When the ball is in the solid state, it rotates with the shell.

(a) Find the period of the oscillations if the ball is always in the solid state.

(b) Find the period of the oscillations if the ball is always in the liquid state.

(c) The ball is now set so that it instantly switches to the liquid state whenever it starts moving downward, and instantly switches to the solid state whenever it starts moving upward. If the initial amplitude of oscillations is θ_0, find the amplitude after n oscillations.