Mechanics VII: Fluids

The fundamental material is covered in chapters 15 and 16 of Halliday, Resnick, and Krane, and at a somewhat higher level in chapter 9 of Wang and Ricardo, volume 1. For a neat explanation of lift and the Coanda effect, see this video. For interesting discussion, see chapters II-40 and II-41 of the Feynman lectures. For a much more advanced introduction which uses vector calculus heavily, see chapters 2–5 and 12–15 of *Physics of Continuous Matter* by Lautrup. There is a total of **86** points.

1 Fluid Statics

Idea 1

In equilibrium, the pressure in a static fluid varies with height as

$$\frac{dp}{dy} = -\rho g.$$

This always holds in equilibrium. For instance, if we squeeze a sealed container of fluid, increasing the pressure locally, then this pressure increase must propagate throughout the entire fluid to maintain $dp/dy = -\rho g$. This is Pascal's principle.

Idea 2: Archimedes' Principle

An object in a fluid experiences an upward buoyant force due to the different pressures on its top and bottom sides. The force is equal in magnitude to the weight of the fluid that would fill the volume of the immersed portion of the object.

This can be surprisingly tricky, so we'll begin with some conceptual questions.

Example 1

A large rock is tied to a balloon filled with air. Both are placed in a lake. As the balloon sinks, how do the air pressure in the balloon, the average density of the balloon, air, and rock system, and magnitude of the net force on the system vary?

Solution

For simplicity, we ignore the elastic force in the balloon itself. Then for the balloon to be in equilibrium, its pressure must match that of the water pressure, so the air pressure in the balloon increases. As the balloon sinks, the rock stays the same volume but the balloon is squeezed smaller, so the density of the system increases. Finally, since the density of water is very approximately constant, the buoyant force on the system is decreasing since its volume is decreasing, so the net force is increasing; the system accelerates downward faster and faster.

[1] **Problem 1** (HRK). The average human body floats in water. SCUBA divers wear weights and a flotation vest that can fill with a varying amount of air to establish neutral buoyancy. A diver is originally neutrally buoyant at a certain depth. How should the diver manipulate the amount of air in their flotation vest to move lower, then stay there at neutral buoyancy?

[2] **Problem 2.** A beaker contains liquid water at its freezing point and has a big ice cube floating in it, also at its freezing point. If the ice cube

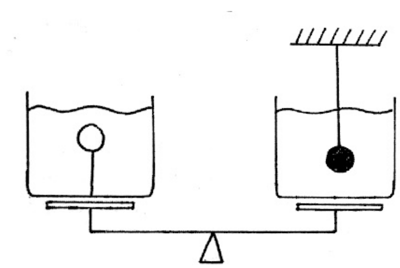
(a) is solid ice,

(b) contains a small metal ball, or

(c) contains a lot of olive oil (which will float on the water in a thin layer),

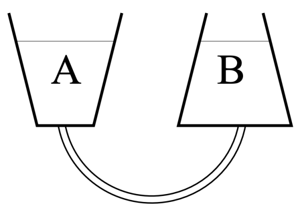
then how does the fluid level change when the cube melts? In all cases, neglect the density of air.

[2] **Problem 3** (Moscow 1939). Consider a pair of scales with identical vessels in which there are equal quantities of water.



In the left-hand vessel you suspend a very light ping-pong ball on a thin, light wire attached to the base of the vessel. In the right-hand vessel you suspend a ping-pong ball filled with lead, again by a light thin wire. Do the scales stay level, go down on the left, or go down on the right?

[2] **Problem 4** (BAUPC). Two trapezoidal containers, connected by a tube as shown, hold water.

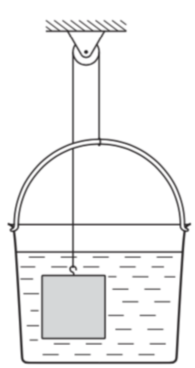


Assume that the containers do not undergo thermal expansion.

(a) If the water in container A is heated, causing it to expand, will water flow through the tube? If so, in which direction?

(b) What if the water in container B is heated instead?

[2] **Problem 5** (MPPP 85). A solid cube of volume V_i and density ρ_i is fastened to one end of a cord, the other end of which is attached to a light bucket containing water, of density $\rho_w = \rho_i/10$.



The system is in equilibrium.

(a) Find the volume V_w of the water in the bucket.

(b) What would happen if more water were poured into the bucket?

(c) What would happen if some or all of the water evaporated?

Example 2

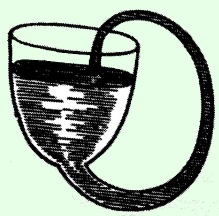
A perfectly spherical, nonrotating planet is covered with water. Geological activity causes a small underwater mountain to form, made of rock that is denser than water. Does the ocean surface above this mountain become higher or lower?

Solution

Systems minimize their energy in equilibrium. This means that in hydrostatic equilibrium, the surface of the water is an equipotential. Since the gravitational field of the mountain increases the gravitational potential near it, the water surface is higher near the mountain.

Example 3

Robert Boyle is best known for Boyle's law, but he also invented a remarkably simple perpetual motion machine, called the *perpetual vase*.



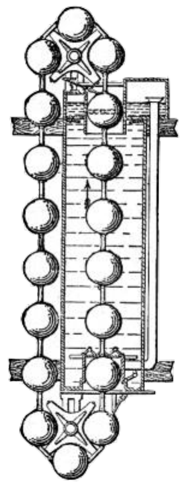
Since the volume of the vase is much greater than the neck, the pressure in the neck cannot possibly hold up all of the water in the vase. Hence the water will flow through the neck and fall back into the vase, causing perpetual motion. Why doesn't this work?

Solution

This is an example of the hydrostatic paradox. Most of the upward force on the water is *not* provided by the pressure in the water in the neck, but from the normal force from the walls; each piece of wall provides enough normal force to hold up all of the water above it. (Of course, ultimately each piece of the glass is held in place by internal forces with other pieces of the glass, which ultimately are balanced by whatever is holding the glass.)

Thus, the water in the neck only supports the water directly above it. That's precisely what is balanced by the heightened pressure in the neck, so the water doesn't start moving. (There have been many more attempts at fluid-based perpetual motion, as you can see here.)

[2] **Problem 6.** Below is another perpetual motion machine, proposed centuries ago.



The balls are less dense than water. The balls on the left are pulled downward by gravity, while the balls on the right are pushed upward by the buoyant force.

(a) Why doesn't this work?

(b) Would it work if the balls and chain were replaced with a flexible tube of constant thickness?

[2] **Problem 7** (HRK)**.** A fluid is rotating at constant angular velocity ω about the vertical axis of a cylindrical container. Defining $z = 0$ to be the water level at the cylinder's axis, show that the liquid surface is the paraboloid

$$z = \frac{\omega^2 r^2}{2g}.$$

Since a paraboloid perfectly focuses incoming light which is parallel to its axis, a rotating fluid can be used as a telescope, as was first pointed out by Isaac Newton. Such liquid-mirror telescopes are cheap, but have the disadvantage that they can only point up. Alternatively, one can gradually cool molten glass in a rotating container so that it solidifies into a paraboloidal lens.

[3] **Problem 8.** USAPhO 2013, problem A4. In order to make measurements, print out the problem before starting.

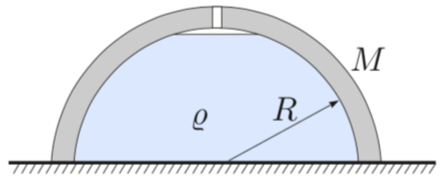
2 Fluid Mechanics

Next we'll consider some situations involving fluids and other objects, where the fluids can be treated at least quasistatically but the objects must be treated dynamically.

Idea 3

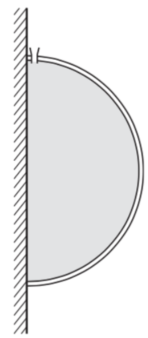
The buoyant force can be regarded as acting at the center of gravity of the fluid displaced by the submerged part of a floating object, called the center of buoyancy. A floating configuration is stable if, when the configuration is slightly rotated, the buoyant force provides a restoring torque about the center of mass.

[2] **Problem 9** (Kalda)**.** A hemispherical container is placed upside-down on a smooth horizontal surface. Water is poured in through a small hole at the bottom of the container. Exactly when the container fills, water starts leaking from between the table and the edge of the container.



Find the mass of the container if the water has density ρ and the hemisphere has radius R.

[2] **Problem 10** (MPPP 89)**.** A thin-walled hemispherical shell of mass m and radius R is pressed against a smooth vertical wall.



It is filled with water through a small aperture at its top, with total mass M. Find the minimum magnitude of the force that has to be applied to the shell to keep the liquid in place.

[3] **Problem 11.** USAPhO 2004, problem A2.

[3] **Problem 12.** USAPhO 2002, problem A4. Be careful with this one!

[3] **Problem 13.** A long log with square cross section and density ρ_l floats in water with density ρ_w. If $\alpha = \rho_l/\rho_w$, then when $\alpha \ll 1$, the log will float stably with one of its sides parallel to the water.

(a) As α is increased, show that once $\alpha > (3-\sqrt{3})/6$, this orientation becomes unstable. (Hint: to keep the calculations short, choose a good coordinate system and work to the lowest relevant order everywhere.)

(b) How do you think the stable orientation of the log varies as α continues to increase? In particular, what it is when $\alpha = 1/2$, or when $\alpha \approx 1$?

Finding the stable orientation of the log for general values of α is quite complicated, but you can play with a nice simulation here; you can also use this to check your answer.

Remark

Some Olympiad questions involving oscillating fluids, which are more subtle. These questions are often impossible to solve exactly, because one must keep track of the entire motion of the water to know how much kinetic and potential energy are in play. In **M4**, you solved IPhO 1984, problem 2, which only asked for an order of magnitude estimate. Physics Cup 2018, problem 4 considers a V-shaped container, where the calculation can be done exactly.

[4] **Problem 14.** EuPhO 2022, problem 1. A nice fluid oscillations problem which can be solved nearly exactly without too much trouble.

Idea 4: Virtual Mass

When an object moves through water, it effectively has extra inertia because it forces water to move as well. This is the "virtual mass" effect (also called added mass, or hydrodynamic mass) which we first mentioned in **M4**. For example, it turns out that

$$\Delta m = \rho \times \begin{cases} (2\pi/3)R^3 & \text{sphere of radius } R \\ \pi R^2 L & \text{cylinder of radius } R, \text{length } L \gg R, \text{moving perpendicular to axis} \\ (8/3)R^3 & \text{thin disc of radius } R, \text{moving along its axis of symmetry} \end{cases}$$

where ρ is the water density. You don't have to memorize these results, but the idea of virtual mass does occasionally show up. For instance, IPhO 1995, problem 3 involves oscillations of a cylindrical buoy of mass m which is only partially submerged in water; they ask you to simply assume a virtual mass $m/3$.

Example 4

Derive the expression for the virtual mass of a sphere.

Solution

Consider a spherical object of radius a moving uniformly with speed v_0 through water of density ρ. The object forces the water to move: the water ahead of it has to get out of the way, while the water behind it needs to fill the space it leaves behind. By the ideas of **M4**, the total kinetic energy of the water is $(\Delta m)v_0^2/2$, where Δm is the virtual mass.

It turns out the fluid's velocity field $\mathbf{v}(\mathbf{r})$ has to satisfy $\nabla\cdot\mathbf{v} = 0$, reflecting the incompressibility of water, and $\nabla \times \mathbf{v} = 0$, reflecting the absence of vorticity. It also has to go to zero far from the sphere, and have zero relative normal velocity at the sphere itself. These differential equations and boundary conditions yield a unique solution. The methods for finding the solution are standard, and typically taught in an undergraduate electromagnetism course, but since they're outside the Olympiad syllabus, I'll just display the answer. The velocity is

$$\mathbf{v}(\mathbf{r}) = \frac{v_0 a^3}{2r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

in polar coordinates, where we placed the origin at the center of the sphere and aligned the $\hat{\mathbf{z}}$ axis with its direction of motion. If you've done **E1**, you might notice this is just like the electric dipole field; this coincidence isn't *too* surprising because that field satisfies the similar equations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$, which are quite restrictive.

Now, to derive the virtual mass, we just have to carry out the kinetic energy integral, which is easiest in spherical coordinates,

$$\begin{aligned} K &= \int \frac{\rho v^2}{2}\, dV \\ &= \frac{\rho v_0^2 a^6}{8} \int_a^\infty \frac{r^2\, dr}{r^6} \int_0^{2\pi} d\phi \int_0^\pi (\sin\theta\, d\theta)(4\cos^2\theta + \sin^2\theta) \\ &= \frac{\rho v_0^2 a^6}{8} \left(\frac{1}{3a^3}\right)(2\pi)(4). \end{aligned}$$

This yields a virtual mass of $(2\pi/3)\rho a^3 = \rho V/2$, as stated above.

Remark

We won't derive the virtual mass for any other shapes, because it tends to require advanced mathematical techniques, outside the Olympiad syllabus. If you're interested in this subject, this paper compiles many exact results, and this paper discusses the history and measurement of virtual mass. Furthermore, Physics Cup 2019, problem 1 and Physics Cup 2024, problem 1 introduce slick methods to calculate virtual mass for some special shapes.

Example 5

What is the initial upward acceleration of a spherical air bubble in water?

Solution

The upward buoyant force on the bubble is $\rho V g$, and the mass of the bubble is negligible, so if we didn't know about virtual mass, we would be tempted to conclude the acceleration is enormous. Instead, the buoyant force is used to move the virtual mass $\rho V/2$ out of the way, so the upward acceleration is $2g$.

Like most things in fluid dynamics, this isn't an exact result. The usual expression for the buoyant force assumes no motion at all, while the virtual mass derivation assumes uniform motion, neither of which are true for an accelerating bubble. For the result above to be accurate, the bubble has to be small, so that the pressure and flow fields have time to reach a quasi-steady state, but not too small, so that we can still ignore viscous forces.

3 Fluid Dynamics

Idea 5: Continuity

In steady flow, the quantity $\rho A v$ is constant along tubes of streamlines.

Idea 6: Bernoulli's Principle

For steady, nonviscous, incompressible flow, the quantity

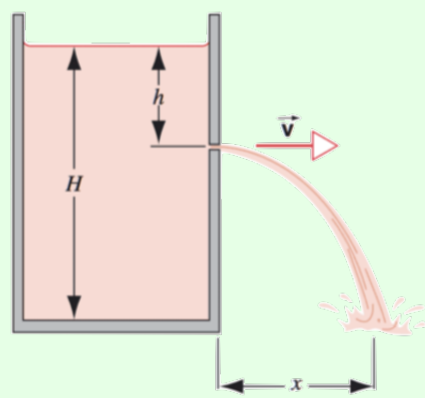
$$p + \frac{1}{2}\rho v^2 + \rho g y$$

is constant along streamlines. Another version of Bernoulli's principle, valid for compressible flow, is given in **T3**. As explained there, the incompressible result here is applicable for water flow, and for gas flow as long as the velocity is much less than the speed of sound.

You might be wondering how steady the flow has to be. Bernoulli's principle is derived by equating work done to kinetic energy, as water flows between two points on a streamline. So you can apply Bernoulli's principle between those two points if the flow is steady on the timescale that it takes fluid to move from one to the other.

Example 6: HRK

A tank is filled with water to a height H. A small hole is punched in one of the walls at a depth h below the water surface as shown.



Find the distance x from the foot of the wall at which the stream strikes the floor.

Solution

The flow isn't perfectly steady, but it's close enough since the hole is small. We thus apply Bernoulli's principle along a streamline, where one point is at the water's top surface, and the other point is just outside the hole. Both points are at atmospheric pressure, because they are directly exposed to the atmosphere. Since the hole is small compared to the tank, the velocity at the first point is small by continuity, so we neglect it, giving

$$\frac{1}{2}\rho v^2 = \rho g h$$

which implies Torricelli's law,

$$v = \sqrt{2gh}.$$

The time t to fall is $t = \sqrt{2(H-h)/g}$, so

$$x = vt = 2\sqrt{h(H-h)}$$

which incidentally is maximized at $h = H/2$.

Incidentally, Bernoulli himself was aware that the answer was different for a large hole, and treated the general case in his 1738 book, *Hydrodynamica*. The method is to apply energy conservation to all of the water at once (i.e. equating the rate of decrease of gravitational potential energy to the rate of increase of total kinetic energy), rather than attempt to apply it along streamlines. You can see this general analysis here.

Example 7

Why should you close your barn door during a storm?

Solution

The wind can flow into the barn, at which point it stops. By Bernoulli's principle, this increases its pressure by $\rho v^2/2$. This creates a net upward force on the roof, which can tear it off the barn.

By the way, even if you do close the barn door, there's a second effect that can still cause a problem: the wind outside has to flow faster along the top to get around it, which decreases its pressure, again creating a net upward force on the roof. This lift effect is very common in real life. You probably already know it's responsible for the lift on an airplane wing. But it also caused my childhood trampoline to achieve liftoff during Hurricane Sandy, destroying a backyard fence. And plumbers rely on it to make sewer pipes "self-clean", by picking up anything stuck to the bottom.

Incidentally, this example brings up a little puzzle about Bernoulli's principle. We argued that the air slows down when it enters the barn, so the pressure goes up. But in the reference frame moving with the wind, the air speeds up when it enters the barn – so shouldn't its pressure go down? The issue with this reasoning is two-fold. First, in the wind's frame, the barn is moving, so the flow isn't steady and Bernoulli's principle doesn't apply. Second, even if the barn were moving slowly, so that the flow were almost steady, the barn's motion would still be doing work on the air, and this changes Bernoulli's principle because it is ultimately a restatement of energy conservation. So in either case, the reasoning fails. When obstacles are present, Bernoulli's principle should always be invoked in the frame of the obstacles.

Example 8: JEE 2020

When a train enters a narrow tunnel, your ears pop because of the pressure change. Find the pressure change, assuming the air has constant density ρ, the atmospheric pressure is P_0, the train speed is v, and the cross-sectional areas of the train and tunnel are A_t and A_0.

Solution

We work in the reference frame of the train. In this frame, the air in the tunnel begins moving towards the train at speed v. When it gets to the train, it has to speed up to speed v_f because it flows through a smaller area $A_0 - A_t$, and this causes its pressure to decrease by Bernoulli's principle. Specifically, we have

$$A_0 v = (A_0 - A_t)v_f, \quad P_f + \frac{1}{2}\rho v_f^2 = P_0 + \frac{1}{2}\rho v^2$$

which gives a pressure drop of

$$P_f - P_0 = -\frac{1}{2}\rho v^2 \left(\frac{1}{(1 - A_t/A_0)^2} - 1\right).$$

We neglected the change in density of the air, which is a good approximation when the train is much slower than the speed of sound. We'll treat fluid flow with changing density in **T3**.

Example 9

A whirly tube is a long, narrow, flexible tube that produces musical tones when swung. Model a whirly tube as a cylinder of length L, rotated about one end with angular velocity ω. For simplicity, neglect gravity. What is the speed of the air when it shoots out the other end?

Solution

The air is slowly sucked from all directions around the entry hole, and shot out at the exit hole. Applying Bernoulli's principle between a point near the entry hole, and the exit hole,

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v_{\text{out}}^2.$$

But that implies $v_{\text{out}} \approx 0$, which doesn't make sense. The problem is that Bernoulli's principle applies to steady flows, and this situation is definitely not steady: by the time the air goes through the tube, the tube has rotated by a significant amount.

Instead, we apply Bernoulli's principle in a reference frame rotating with the tube. The centrifugal force gives an additional term, turning it into

$$P + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 r^2 = \text{const}.$$

Applying Bernoulli's principle between the same two points gives

$$P_{\text{atm}} \approx P_{\text{atm}} + \frac{1}{2}\rho v^2 - \frac{1}{2}\rho\omega^2 L^2$$

from which we conclude $v = \omega L$. Transforming back to the original reference frame, the exit speed of the air is $\sqrt{v^2 + (\omega L)^2} = \sqrt{2}\, v$.

Example 10

A big fan produces a stream of air with speed v. If the atmospheric pressure in the room is P_{atm}, what's the pressure P in the middle of the fan's air stream?

Solution

This question frequently appears in middle school physics lessons. Obviously, if we apply Bernoulli's principle to the air before and after it goes through the fan, we get

$$P + \frac{1}{2}\rho v^2 = P_{\text{atm}}$$

so that the pressure is lower than atmospheric pressure. Easy, right? But it's wrong!

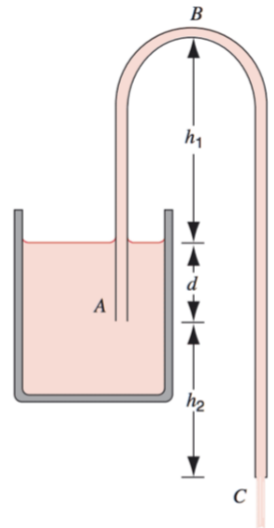
The air in the stream is traveling forward with constant velocity, exposed to the rest of the air in the room, which has atmospheric pressure. If there actually was such a pressure difference, the fan's air stream would be compressed by the air in the room, until it reached atmospheric pressure again. If you look back carefully at the above examples, you'll see this is always the case: air can only be at a different pressure if it's confined away from the atmosphere at large (e.g. in a train tunnel or a whirly tube), or if it's actively being accelerated (e.g. when it flies into or over a barn, in which case the pressure difference is precisely what causes the force). The other case where you can maintain a pressure difference is when the air is moving extremely quickly, which will be discussed in **T3**.

So the correct answer is that $P = P_{\text{atm}}$. But why doesn't Bernoulli's principle work? Because it's a statement of energy conservation, and the fan itself is doing work on the air to get it moving. The correct statement would be

$$P_{\text{atm}} + \frac{1}{2}\rho v^2 = P_{\text{atm}} + w$$

where w is the work done by the fan per unit volume of air.

[2] **Problem 15** (HRK)**.** A siphon is a device for removing liquid from a container that cannot be tipped. An example of a siphon, with constant cross-section, is shown below.



The tube must initially be filled, but once this has been done the liquid will flow until its level drops below the tube opening at A. The liquid has density ρ and negligible viscosity.

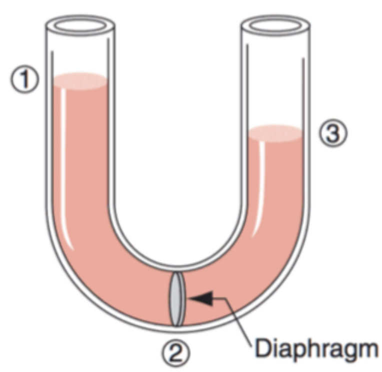
(a) With what speed does the liquid emerge from the tube at C?

(b) What is the pressure of the liquid at the topmost point B?

(c) What is the maximum possible h_1 so that the siphon can operate?

(d) Would the siphon still work if h_2 were slightly negative? How negative can it be, for the siphon to keep on working?

[2] **Problem 16** (HRK)**.** Consider a uniform U-tube with a diaphragm shown below.



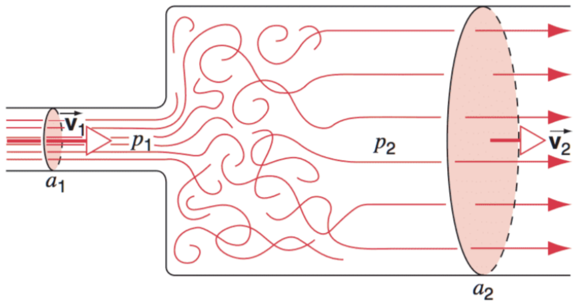
(a) Suppose the diaphragm is opened and the liquid begins to flow from left to right. Show that applying Bernoulli's principle yields a contradiction.

(b) Explain why Bernoulli's principle doesn't apply if the diaphragm has a very wide opening.

(c) Explain why Bernoulli's principle doesn't apply if the diaphragm has a tiny opening.

For a similar idea to this problem, see $F = ma$ 2018 A22.

[2] **Problem 17** (HRK). A stream of fluid of density ρ with speed v_1 passes abruptly from a cylindrical pipe of cross-sectional area a_1 into a wider cylindrical pipe of cross-sectional area a_2 as shown.



The jet will mix with the surrounding fluid, forming a turbulent region where the pressure is approximately p_1. Further to the right, the flow becomes almost uniform again, with average speed v_2 and pressure p_2.

(a) By considering force and momentum, show that

$$p_2 - p_1 = \rho v_2(v_1 - v_2).$$

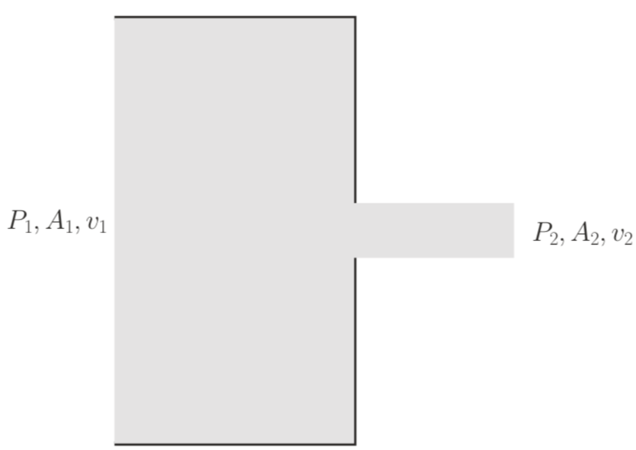
(b) Show from Bernoulli's principle that in a gradually widening pipe we would instead get

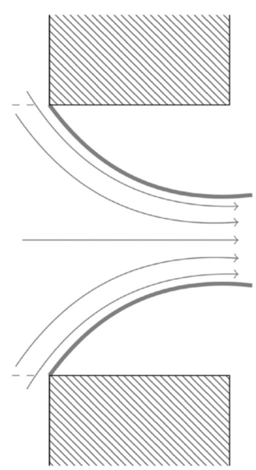
$$p_2 - p_1 = \frac{1}{2}\rho(v_1^2 - v_2^2).$$

(c) Find the loss of pressure due to the abrupt enlargement of the pipe. Can you draw an analogy with elastic and inelastic collisions in particle mechanics?

[2] **Problem 18** (PPP 49). After a tap above an empty rectangular basin has been opened, the basin fills with water in a time T_1. After the tap has been closed, opening a plug at the bottom of the basin empties it in time T_2. If both the tap and plug are open, what ratio of T_1/T_2 can cause the basin to overflow?

[4] **Problem 19.** This problem is about the subtle phenomenon of *vena contracta*. An incompressible fluid of density ρ is flowing through a tube of area A_1, which suddenly contracts to area $A_2 \ll A_1$. Naively, the flow looks as shown at left below.





(a) Argue by energy conservation that $v_2 \approx \sqrt{2(P_1 - P_2)/\rho}$.

(b) Argue that the net force on the fluid shown in the picture is approximately $(P_1 - P_2)A_2$. Then argue by momentum conservation that $v_2 \approx \sqrt{(P_1 - P_2)/\rho}$.

(c) The resolution of the paradox is that in part (a), we're actually solving for the final speed of the water, and in part (b), we're actually solving for the horizontal component of the velocity. So the resolution has to be that the fluid does *not* exit through the orifice purely horizontally. Instead, it contracts as it exits, as shown at right above, eventually shrinking to a minimum area A_3, at which point the flow actually is horizontal. Assume for simplicity that $P_1 \gg P_3$. Show that the final area is $A_3 \approx A_2/2$, so that the hole is effectively only half its size.

(d) Even assuming ideal fluid flow satisfying Bernoulli's principle, the result above for A_3 is not exact, but is instead off by about 20% for the sudden opening shown above. Is the true value of A_3 higher or lower than $A_2/2$?

(e) How could the shape of the orifice be modified so that A_3 is almost exactly $A_2/2$? How could the orifice be modified so that the water comes out perfectly straight?

Remark

Vena contracta is too subtle for introductory textbooks, but it makes a big practical difference. For example, if you estimate how long it takes water in a bucket to empty through a hole using Torricelli's law, you'll be off by about a factor of 2 if you don't include vena contracta! And Halliday, Resnick, and Krane don't consider it in their example titled "thrust on a rocket", getting a thrust which is also off by a factor of 2. Of course, real plumbers and rocket scientists are perfectly aware of vena contracta, and carefully design nozzles and drains to account for it. For further discussion, see this paper.

4 Fluid Systems

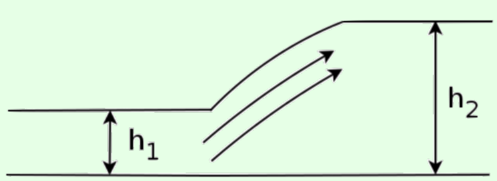
Now we put it all together and consider complex mechanical systems with moving fluids.

Idea 7

If a fluid is moving in a complex way, it's usually difficult to say anything by directly considering the flow. Instead, it's easier to apply conservation laws.

Example 11

A fluid of density ρ flowing with a fast velocity v_1 and height h_1 can undergo a "hydraulic jump", where the height of the fluid increases to h_2. At the same time, the fluid flow slows down and becomes turbulent.



This phenomenon is very common in everyday life. For example, it happens whenever you turn on the water faucet in a sink; the hydraulic jump occurs on a circle centered on the faucet. Find the final height h_2.

Solution

During this process, the bulk kinetic energy of the water is not conserved, because it is converted to turbulent motion. However, the horizontal momentum of the water is approximately conserved. Consider a stream of water of width w flowing in the x direction, where the hydraulic jump occurs at $x = 0$. By mass conservation,

$$v_1 h_1 = v_2 h_2$$

where v_2 is the final speed. Now we consider a fixed subset of the water encompassing the hydraulic jump. The atmospheric pressure does not yield a net horizontal force on the water, so we focus on the pressure in excess of atmospheric pressure. The total excess pressure force on the left end is

$$F_\ell = \int_0^{h_1} \rho g h w \, dh = \frac{1}{2} \rho g w h_1^2.$$

Therefore, we have total force

$$F = \frac{1}{2} \rho g w (h_1^2 - h_2^2).$$

On the other hand, the mass of water that flows through the hydraulic jump per unit time is $\rho h_1 w v_1$, and its velocity decreases by $v_1 - v_2$, so

$$\frac{dp}{dt} = \rho h_1 w v_1 (v_1 - v_2) = \rho w v_1 v_2 (h_2 - h_1)$$

where we used mass conservation. Equating $F = dp/dt$ and simplifying gives

$$g(h_1 + h_2) = 2 v_1 v_2.$$

Applying mass conservation again leads to a quadratic in h_2,

$$h_2^2 + h_1 h_2 - \frac{2 v_1^2 h_1}{g} = 0$$

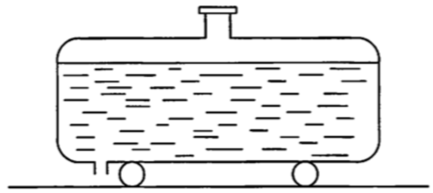
and the physically relevant positive solution is the answer,

$$h_2 = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2 h_1 v_1^2}{g}}.$$

For $v_1^2 > g h_1$, we have $h_2 > h_1$ and an ordinary hydraulic jump. For $v_1^2 < g h_1$, you might expect a "reverse" hydraulic jump to occur, but this is impossible by the second law of thermodynamics. In a hydraulic jump, some of the kinetic energy of laminar flow energy is converted to turbulent flow, which is essentially heat; thus the reverse can't happen. So in addition to deriving h_2, we've found the minimum v_1 for a hydraulic jump to be possible!

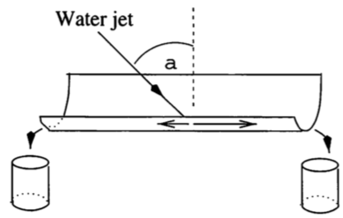
Note that this conservation law approach doesn't tell us about how far a fluid will flow before it undergoes a hydraulic jump. That would require understanding the fluid flow in detail, accounting for turbulence and viscosity, which is generally analytically intractable. For more on this subject, see sections 26.1 and 26.2 of Lautrup.

[3] **Problem 20** (PPP 70). A tanker full of liquid is at rest on a frictionless horizontal road.



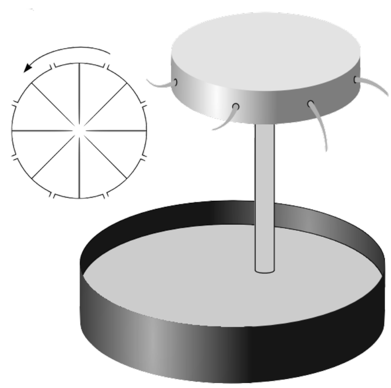
A small vertical outlet pipe at the rear of the tanker is opened. Describe qualitatively how the tanker will move (a) immediately afterward, and (b) after a long time. Assume that the water always falls out of the cart with zero horizontal velocity in the cart's frame.

[3] **Problem 21** (PPP 74). A jet of water strikes a horizontal gutter of semicircular cross-section obliquely, as shown.



The jet lies in the vertical plane that contains the center-line of the gutter. Assume the angle is relatively shallow, so that the water hits the gutter smoothly, and doesn't splatter. Find the ratio of the quantities of water flowing out at the two ends of the gutter as a function of the angle of incidence α of the jet.

[3] **Problem 22** (EFPhO 2005). A water pump consists of a vertical tube of cross-sectional area S_1 topped with a cylindrical rotating tank of radius r. All the vessels are filled with water; there are holes of total cross-sectional area $S_2 \ll S_1$ along the perimeter of the tank, which are open for the operating regime of the pump. The height of the tank from the water surface of the reservoir is h. An electric engine keeps the vessel rotation at angular velocity ω. The water density is ρ, the atmospheric pressure is p_0, and the saturated vapor pressure is p_k. Inside the tank there are metal blades, which make the water rotate with the tank.



(a) Find the pressure p_2 at the perimeter of the tank when all the holes are closed.

(b) For the rest of the problem, we suppose the holes are opened. Find the velocity v_2 of the water jets with respect to the ground.

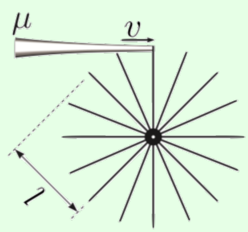
(c) If the tank rotates too fast, the water pressure at some point will become lower than p_k. As you'll see in **T3**, this will cause "cavitation", i.e. the water will start boiling, lowering the pump's efficiency. Find the highest cavitation-free angular speed ω_{max}.

(d) If the power of the electric engine is P, what is the theoretical upper limit of the volume pumped per unit time, assuming S_2 can be freely adjusted?

[3] **Problem 23.** A helicopter with length scale ℓ and density ρ_h can hover using power P, in air of density ρ_a. Find a rough estimate for P in terms of the given parameters. (For a nice followup discussion of lift, see section 3.6 of The Art of Insight.)

Example 12: Kalda 82

A water turbine consists of a large number of paddles that could be considered as light flat boards with length ℓ, that are at one end attached to a rotating axis. The paddles' free ends are positions on the surface of an imaginary cylinder that is coaxial with the turbine's axis. A stream of water with velocity v and flow rate μ (kg/s) is directed on the turbine such that it only hits the edges of the paddles.



Find the maximum possible power that can be extracted.

Solution

Let v_t be the speed of the edge of the turbine. In time dt, the amount of mass of water that collides with the turbine is

$$dm = \frac{\mu}{v}(v - v_t)\,dt.$$

The horizontal force on the paddle is

$$F = \frac{dp}{dt} = \frac{dm}{dt}\Delta v = \frac{\mu}{v}(v - v_t)^2$$

so the power delivered to the turbine is

$$P = Fv_t = \frac{\mu v_t}{v}(v - v_t)^2.$$

Maximizing this by setting $dP/dv_t = 0$ gives $v_t = v/3$, so the maximum power is $4\mu v^2/27$. This is 8/27 of the total power in the incoming water.

[3] **Problem 24.** Air of constant density ρ and wind speed v_i is heading directly towards a windmill of area A. When the wind gets to the windmill blades, it is traveling forward with speed v_f. Well after it leaves the vicinity of the blades, it has speed v_o. The design of the windmill, such as the shape and speed at which its blades turn, can be adjusted to set the value of v_f.

(a) Find the power going from the wind to the turbine by using energy conservation, assuming that there are no extraneous energy losses, e.g. to turbulence.

(b) Find the power going from the wind to the turbine by considering the force of the windmill on the air and using momentum conservation, again assuming no extraneous energy losses.

(c) Find an upper bound on the ratio of the wind power that can be harvested by the windmill, to the amount of wind power that would pass through it if it weren't running.

This result is called the Betz limit.

[5] **Problem 25.** GPhO 2017, problem 2. A very tricky composite fluids/mechanics problem.

5 Wet Water

So far we've mostly ignored viscosity and turbulence, an unrealistic limit that some refer to as "dry water". Now we'll consider some problems involving real, wet water.

Idea 8

A velocity gradient is associated with a drag force

$$F = \eta A \frac{dv}{dy}$$

where η is the (dynamic) viscosity. In addition, for fluid flowing next to a wall, the layer of fluid right next to the wall is approximately at rest.

Example 13: HRK

Prairie dogs live in large colonies in complex interconnected burrow systems. They face the problem of maintaining a sufficient air supply to their burrows to avoid suffocation. They avoid this by building conical earth mounds about some of their many burrow openings. How does this air conditioning scheme work?

Solution

Because of viscous effects, the wind speed is small near the ground, and hence grows with height. By Bernoulli's principle, this means the pressure at the top of a mound is slightly lower than the pressure at an opening without a mound. This difference in pressure drives air flow through the burrows.

Example 14

If you've used a standard garden hose, you might have noticed that the water shoots higher if you partially block the outlet with your finger. Why does this happen?

Solution

The water company provides water to your house at a fixed pressure $P_{\text{atm}} + \Delta P$. Thus, naively the water should always shoot equally far, because Bernoulli's principle says the exit speed is $v = \sqrt{2\Delta P/\rho}$, corresponding to a peak height $\Delta P/\rho g$, independent of the area of the hole. (There is a vena contracta effect, as mentioned in problem 19, but this also doesn't depend on the area.)

The resolution is that for a typical long, thin garden hose, viscous losses dominate. As you'll see in problem 26, a higher mass flow rate leads to a higher drop in pressure. When you partially block the outlet, you're simply decreasing the flow rate, so that viscosity has a smaller effect, allowing the water to get closer to the maximum possible height $\Delta P/\rho g$.

In plumbing, the quantity $\Delta P/\rho g$ is called the "pressure head", and effects like viscosity give rise to "head loss". Unfortunately, for most realistic pipes it is intractable to calculate the head loss, because the water flow is turbulent. Instead, the amount of head loss is parametrized by the so-called Darcy friction factor, whose values are tabulated in references.

Example 15

If you stir a cup of coffee, around how long does it take the rotational motion to settle down?

Solution

The rotational motion stops because of viscous drag against the walls. For concreteness, let's suppose the coffee has density ρ, viscosity η, and is in a mug of radius R and height $H \gg R$ (so most of the drag comes from the vertical wall of the mug). The angular momentum is

$$L \sim I\omega \sim \rho R^4 H\omega.$$

The damping torque due to viscous forces is

$$\tau \sim RF \sim \eta A \frac{dv}{dr} R$$

and since the drag is from the vertical wall, $A \sim HR$. Estimating the velocity gradient dv/dr is a little trickier. As mentioned above, the coffee *right* next to the wall has zero velocity, while the coffee slightly inward from the wall has speed $v \sim R\omega$. The velocity transitions between these two values in a thin "boundary layer".

Finding the exact thickness of this boundary layer would require solving complicated differential equations, but it suffices to use dimensional analysis. Note that R and H can't possibly play a role, since the layer is so thin it doesn't "see" the shape of the mug. The fluid properties η and ρ surely matter. Perhaps more subtly, ω matters. If the fluid weren't spinning, but rather were uniformly translating in a plane, then the boundary layer would just grow over time until it was the size of the whole fluid. That's what we saw in problem 26, where the velocity changes gradually along the whole pipe radius R. The boundary layer doesn't grow to the whole mug's size here, because the velocity it's trying to match is constantly changing over the timescale $1/\omega$.

Using dimensional analysis, we thus conclude the boundary layer has thickness

$$\Delta r \sim \sqrt{\frac{\eta}{\rho\omega}}.$$

The damping torque is

$$\tau \sim \eta\,(HR)\frac{R\omega}{\Delta r}R \sim \sqrt{\rho\eta\omega^3}\,HR^3$$

so the timescale for damping is

$$T \sim \frac{L}{\tau} \sim \sqrt{\frac{\rho}{\eta\omega}}\,R.$$

Numerically, if we use the rough estimates

$$\rho \sim 10^3\,\mathrm{kg/m^3}, \quad \omega \sim 10\,\mathrm{s^{-1}}, \quad R \sim 0.1\,\mathrm{m}, \quad \eta \sim 10^{-3}\,\mathrm{Pa\,s}$$

where η is the value for room temperature water, then we get the reasonable results

$$\Delta r \sim 0.3\,\mathrm{mm}, \quad T \sim 30\,\mathrm{s}.$$

[3] **Problem 26.** Water flows through a cylindrical pipe of radius R and length $L \gg R$, across which a pressure difference Δp is applied.

(a) If the flow is slow, viscous effects dominate. By balancing forces on a cylinder of fluid, show that

$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2).$$

Then show that the total mass flux is

$$\frac{dm}{dt} = \frac{\rho\pi R^4 \Delta p}{8\eta L}.$$

This is called Poiseuille's law.

(b) If the flow is very fast, the flow is turbulent. Viscous effects are negligible, and the work done by the pressure difference is dissipated by turbulence into internal energy. Find a rough estimate of the mass flow rate.

[4] **Problem 27.** When a spherical object of radius R moves with velocity v through a fluid of viscosity η and density ρ, it experiences a drag force.

(a) Apply dimensional analysis to constrain the possible forms of the drag force F. You should find there is one dimensionless quantity inversely proportional to η, in accordance with the Buckingham Pi theorem of **P1**. This dimensionless quantity is called the Reynolds number, and it determines what kind of drag dominates.

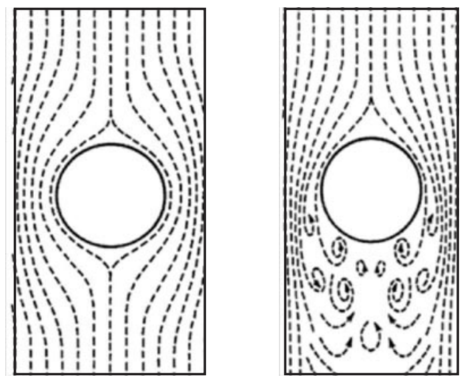
(b) It turns out that $F \propto v$ at low velocities and $F \propto v^2$ at high velocities. Using this information, find the form of the drag force in both cases. (For reference below: the answers are

$$F = 6\pi\eta R v, \quad F = \frac{1}{2}C_d \rho A v^2$$

where C_d is a dimensionless drag coefficient, which is about 1/2 for a sphere. The drag coefficient depends strongly on the shape of the object, being much smaller for streamlined shapes, and weakly on the velocity.)

(c) Hot water has density $\rho = 10^3\,\mathrm{kg/m^3}$ and viscosity $\eta = 0.3 \times 10^{-3}\,\mathrm{Pa\,s}$. (Room temperature water has about 3 times the viscosity.) For an object of radius 1 cm, find the characteristic velocity that divides the two types of drag.

(d) The two cases correspond to flow patterns as shown below.



In the latter case, a region of turbulent flow is created. Using this picture, explain why the drag force is proportional to v^2.

(e) The results above apply to both liquids and gases. In a gas, the relevant quantities are the mass m of the gas molecules, their typical speed u, their number density n, and radius r (which determines how often their collide with each other). Use dimensional analysis to constrain the possible forms of the viscosity η. How do you think η scales with n?

Drag is nicely discussed throughout The Art of Insight; see sections 3.5, 5.3.2, and 8.3.1.2.

Remark

Without knowing the answer to part (b) above, one might expect that the drag force can depend on η, ρ, v, and the shape of the object. In the linear case, the drag force does not depend on ρ. In the quadratic case, the drag force does not depend on η.

These differences can be understood by thinking of where the energy dissipated is going. In the quadratic case, the fluid picks up macroscopic kinetic energy, in the form of a turbulent flow pattern, which is why the drag force does not depend on η. In the linear case, the fluid slows smoothly and hence does not pick up any macroscopic energy; instead the energy is dissipated as heat. Since the macroscopic kinetic energy is not involved, the drag force does not depend on ρ. (Of course, in the quadratic case the turbulent motion eventually stops; at this point it has been converted to heat. The time it takes this to happen is set by η, but it occurs well after the object has passed by and hence does not affect the drag force.)

Example 16

If raindrops fall, why don't clouds fall?

Solution

This isn't a stupid question! It's actually a tough one, which stumped the ancient Greeks and Romans. To give context, we'll cover a bit of atmospheric physics, a topic we will continue in **T1** and **T3**. This is all a bit of a simplification of an interesting story, told in more detail in chapter II-9 of the Feynman lectures.

First, it's useful to review the water cycle. Sunlight directly warms up the ground, and the ground thereby warms the air near the ground. Since warmer air at the same pressure is less dense, it begins to rise by convection. This air also expands roughly adiabatically as it rises, lowering its temperature. Warmer air can also hold more water, so if the original air was moist, water vapor will condense into droplets as the air rises. (This last point is important, because the condensation releases energy, partially counteracting the cooling of the rising air. This keeps it warmer and hence lighter than its surroundings, allowing it to continue to rise.)

Now consider a droplet of radius r. Depending on the droplet size and velocity, the drag force scales as r or r^2, while the gravitational force scales as r^3. Hence the tiny water droplets in clouds are hence carried upward with the ascending moist air, since the drag force dominates. They fall down once they accrete into sufficiently large raindrops, where gravity dominates.

Incidentally, falling raindrops do not have the teardrop shape shown in typical illustrations. Small raindrops are nearly spherical, because of surface tension. Large raindrops are squashed by air resistance into a "hamburger" shape.

Example 17

Why can you see through both humid air and heavy rain, but not through fog or a cloud, which contains droplets of intermediate size?

Solution

Let's consider a fixed number of water molecules in a fixed volume. When they're all separated, we have humid air. As the molecules join into small droplets (say, with $n \lesssim 100$ molecules each) the amount of electromagnetic radiation scattered by each droplet grows as n^2 because of constructive interference (as discussed in **E7**), which allows them to scatter a larger fraction of the light that passes through them.

But for the very large droplets found in rain, the trend turns around. These droplets are much larger than the wavelength of light, which means that we're in the geometric optics limit. They can scatter at most 100% of the light that falls on them, which scales as their area. Since the volume goes as n, the area goes as $n^{2/3}$.

We therefore conclude that

$$\frac{\text{scattering}}{\text{water molecule}} \sim \begin{cases} n & \text{small droplets in cloud/fog} \\ n^{-1/3} & \text{large droplets in rain} \end{cases}$$

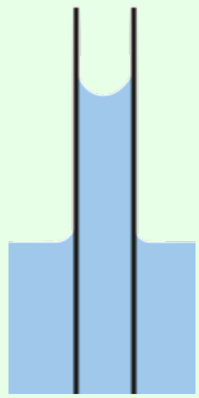
so that clouds occupy a sweet spot, scattering the most light for a given amount of water. The same applies for fog, which is simply a cloud that touches the ground.

6 Surface Tension

We now return to surface tension, first covered in **M2**, which we'll see yet again in **T3**.

Example 18

A very thin, hollow glass tube of radius r is dipped vertically inside a container of water.



Find the height to which water can climb in the tube.

Solution

In **M2**, we considered problems that could be solved knowing only the "surface tension of water" γ, which is the energy cost per unit area of having a water-air interface. But in this problem there is also a water-glass interface, and the answer to the question depends on precisely how water and glass interact. Specifically, you need to know the surface tension coefficient γ_{wg} which determines the energy cost of having a water-glass interface.

Fortunately, it turns out you don't need to know γ_{wg} if you know the contact angle θ, i.e. the angle between the glass and water surface at the top of the meniscus, which is drawn as acute in the diagram above. We'll just treat θ as a given, but for an explanation of how θ is determined, see **T3** or section 5.5 of Lautrup.

Since the glass tube is very thin, surface tension determines the shape of the water-air surface, so it is spherical since spheres minimize area. By some elementary geometry, one can show that the radius of curvature of this sphere is $R = r/\cos\theta$.

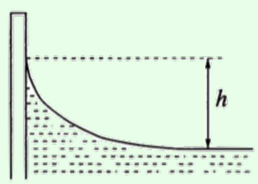
We showed using force balance arguments in **M2** that the pressure inside the curved water surface is lower than atmospheric pressure by $\Delta P = 2\gamma/R$. On the other hand, we also know from Pascal's principle that $\Delta P = \rho g h$. Equating the two gives

$$h = \frac{2\gamma \cos\theta}{\rho g r}.$$

This is Jurin's law. Ideally, water and glass have zero contact angle. This implies that water perfectly wets glass, i.e. that a droplet of water placed on a horizontal glass surface will spread to cover it completely (though this doesn't happen in reality because glass tends to quickly get coated in a layer of impurities). Making this assumption, which we will use for problems below, the answer reduces to $h = 2\gamma/\rho g r$.

Example 19: PPP 130

Water in a glass beaker forms a meniscus, as shown below.



Find the height h to which the meniscus rises above the flat water surface.

Solution

We consider all of the external horizontal forces acting on the water. The surface tension force acting at the top of the meniscus is purely vertical, because water and glass have zero contact angle. The other surface tension force acting on the flat part of the water is γ per length. This balances the excess hydrostatic pressure (i.e. the pressure above atmospheric pressure) at the wall, which is $\rho g h^2/2$ per unit length. Thus,

$$h = \sqrt{\frac{2\gamma}{\rho g}}.$$

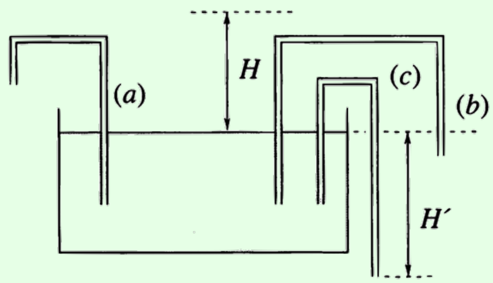
We could have also gotten this with dimensional analysis, up to the prefactor.

Remark

You might be wondering how to compute the shape of the meniscus. There are two methods. First, the pressure right above the water surface is P_{atm}, so the pressure right below the water surface can be determined from the radii of curvature of the surface, using the Young–Laplace equation from **M2**. This pressure can also be computed from the height of the surface using Pascal's principle. Combining these two yields a differential equation for the shape with a rather complicated solution, as explained in sections 5.6 and 5.7 of Lautrup. As you'll see in problem 32, you can also derive this result by considering force balance on the water.

Example 20: PPP 29

Water can rise to a height H in a certain capillary tube. Three "gallows" are made from this tubing by bending it, and placed into a tank of water.



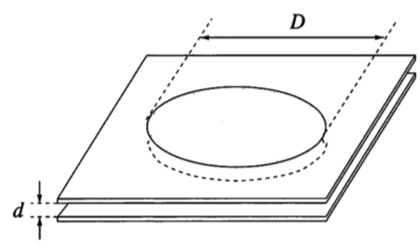
Note that $H' > H$. For which tubes, if any, does water flow out of the other end?

Solution

Clearly no water can fall out of (a), because this would produce a perpetual motion machine. The gallows (b) and (c) are a bit more subtle. Water will *not* fall out of a capillary tube if its end is less than a height H below the free water surface; this follows from the same derivation as Jurin's law, with the surface tension acting to hold the water in the tube. So water only falls out of (c).

[2] **Problem 28.** A soap bubble of radius R and surface tension γ has a small tube of radius $r \ll R$ passing through its surface. If the air has density ρ, find the rate of decrease of R.

[2] **Problem 29** (PPP 63)**.** Water is stuck between two parallel glass plates. The distance between the plates is d, and the diameter of the trapped water disc is $D \gg d$.



In terms of the surface tension γ of water, what is the force acting between the two plates? This effect can cause wet glass plates to stick together.

[3] **Problem 30** (EFPhO 2009)**.** A soap film of thickness $h = 1\,\mu$m is formed inside a ring of diameter $D = 10\,$cm, and the surface tension of the film is $\gamma = 0.025\,$N/m. If the film is broken at the center, it will begin to fall apart; estimate the time needed for this to happen.

[3] **Problem 31.** USAPhO 2020, problem B1. A nice, slightly mathematically involved surface tension problem with a real-world impact. This setup is discussed in detail in section 5.4 of Lautrup.

[4] **Problem 32.** IPhO 2023, problem 3, parts B and C. A nice problem on the shape of a meniscus, which also explains why pieces of cereal attract each other in a bowl of milk.

Example 21: IPhO 2022 3B

Slightly wet sand is much stronger than either dry sand or very wet sand, which allows the construction of large structures like sand castles. Why is this, and how does the strength depend on the typical size r of the sand grains?

Solution

When a pile of sand is dry, the only force keeping it in place is friction, which is weak. When it's very wet, it's essentially just water, which will simply collapse. But when it's slightly wet, adjacent sand grains have a small layer of water connecting them. Since sand grains are small, this implies a huge total surface area, and thus large surface tension effects.

There are actually two conceptually distinct components to the effect. First, the bit of water connecting two sand grains will provide a surface tension force $F \sim \gamma r$. Second, as you saw in problem 29, the water has a pressure lower by $\Delta P \sim \gamma / r$, leading to an attractive pressure force $(\Delta P)A \sim \gamma r$. In either case, that means the force needed to displace a single grain of sand scales with r. The number of sand grains in a fixed cross-sectional area scales as $1/r^2$, so the weight a sand castle can bear scales as $1/r$. Thus, fine-grained sand is stronger.

This is another example of the subtleties of granular media, first mentioned in **M2**. Neither sand nor water are strong on their own, but they're strong together. Water provides the forces, while the sand provide the structure which lets those forces be effective.