Lecture Notes on

Statistical Field Theory

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These notes cover statistical field theory and the renormalization group. The primary sources were:

- Kardar, *Statistical Physics of Fields*. A concise and logically tight presentation of the subject, with good problems. Possibly a bit too terse unless paired with the 8.334 video lectures.
- David Tong's Statistical Field Theory lecture notes. A readable, easygoing introduction covering the core material of Kardar's book, written to seamlessly pair with a standard course in quantum field theory.
- Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*. Covers similar material to Kardar's book with a conversational tone, focusing on the conceptual basis for phase transitions and motivation for the renormalization group.

The notes are structured around the MIT course based on Kardar's textbook, and were revised to include material from Part III *Statistical Field Theory* as lectured in 2017. Sections containing this additional material are marked with stars. The most recent version is here; please report any errors found to kzhou7@gmail.com.

Contents

1 Introduction

We've learned to use statistical mechanics to go between microscopic properties of matter (i.e. the degrees of freedom for each particle) and macroscopic thermodynamic variables, such as pressure and temperature. We were able to handle noninteracting systems exactly, and account for weak interactions using perturbation theory. In statistical field theory, we investigate situations where the perturbative approach breaks down, often at a phase transition. We would like to answer the following questions:

1. In the thermodynamic limit, how do distinct macroscopic phases emerge from microscopic dynamics? What are the thermodynamic variables that describe the macroscopic state of these phases, and what are their identifying signatures in the measurement of bulk response functions (e.g. heat capacity, susceptibility)?

2. What are the characteristic low energy excitations of the system? In many cases, such as phonons in solids or superfluid helium, they are collective modes which involve the motion of many microscopic degrees of freedom. These low-lying modes are easily excited by thermal fluctuations, and probed by scattering experiments.

In general, a true *ab initio* calculation is intractable. However, in some cases, the description may be simplified by averaging over many particles in the thermodynamic limit. At long wavelengths, we may coarse-grain the system into a field (the 'hydrodynamic limit'), and apply classical field theory techniques.

1.1 Phonons

As a first example, we calculate the phonon contribution to the specific heat capacity of a solid using both approaches. First, consider an *ab initio* approach. The steps are as follows.

- Starting with the Schrodinger equation, we perform approximations (i.e. density functional theory) to yield a many-body potential $V(\mathbf{q}_1, \ldots, \mathbf{q}_N)$.

- We find the lattice ion positions at zero temperature by minimizing V. Typically, we find a Bravais lattice with

$$\mathbf{q}^*(l, m, n) = l\mathbf{a} + m\mathbf{b} + n\mathbf{c} = \mathbf{q}^*_{\mathbf{r}}, \quad \mathbf{r} = (l, m, n).$$

- We expand the potential to second order about this minimum. If the displacements are $\mathbf{u}(\mathbf{r})$,

$$\Delta V = \frac{1}{2} \sum \frac{\partial^2 V}{\partial q_{\mathbf{r},\alpha} \partial q_{\mathbf{r},\beta}} u_\alpha(\mathbf{r}) u_\beta(\mathbf{r})$$

  where α and β are spatial components. The full Hamiltonian for small deformations is found by adding $\sum p_\alpha(\mathbf{r})^2/2m$ where $p_\alpha(\mathbf{r})$ is the conjugate momentum to $u_\alpha(\mathbf{r})$.

- By applying translational symmetry,

$$\frac{\partial^2 V}{\partial q_{\mathbf{r},\alpha} \partial q_{\mathbf{r},\beta}} = K_{\alpha\beta}(\mathbf{r} - \mathbf{r}')$$

the Fourier transform of the Hamiltonian becomes

$$H = \sum_{\mathbf{k},\alpha,\beta} \left(\frac{p_\alpha(\mathbf{k})^2}{m} + K_{\alpha\beta}(\mathbf{k}) u_\alpha(\mathbf{k}) u_\beta(\mathbf{k})^* \right)$$

where the sum in $\mathbf{k}$ is over the Brillouin zone.

- The matrix $K_{\alpha\beta}(\mathbf{k})$ can now be diagonalized separately for each $\mathbf{k}$, which can be done with the help of the crystal symmetry. This decomposes the Hamiltonian into $3N$ independent harmonic oscillators.
- Quantizing each oscillator, we find

$$H = \sum_{\mathbf{k},\alpha} \hbar\omega_\alpha(\mathbf{k}) \left(n_\alpha(\mathbf{k}) + \frac{1}{2} \right)$$

where the $\omega_\alpha(\mathbf{k})$ are the three normal mode frequencies for each $\mathbf{k}$.

- Therefore the average energy at a temperature T is given by

$$E(T) = \sum_{\mathbf{k},\alpha} \hbar\omega_\alpha(\mathbf{k}) \left(\langle n_\alpha(\mathbf{k}) \rangle + \frac{1}{2} \right), \quad \langle n_\alpha(\mathbf{k}) \rangle = \frac{1}{\exp(\hbar\omega_\alpha / kT) - 1}$$

where we applied the Bose–Einstein distribution. We see that $E(T)$ has complex behavior, dependent on microscopic details through the ω_α.

However, it is possible to extract universal behavior in the long-wavelength limit. To see this, we switch to a chain of particles in one dimension.

- The most general quadratic translationally invariant potential energy for deformations is

$$V = \frac{K_1}{2} \sum_n (u_{n+1} - u_n)^2 + \frac{K_2}{2} \sum_n (u_{n+2} - u_n)^2 + \cdots .$$

- The normal modes are just the Fourier components, with

$$u_n = \int_{-\pi/a}^{\pi/a} đk \, e^{-ikna} u(k), \quad u(k) = a \sum_n e^{ikna} u_n$$

where a is the equilibrium separation between the particles.

- Plugging in the Fourier transform, doing the sum over n to get a delta function, and using $u(-k) = u^*(k)$, which follows from u_n being real, we have

$$V \propto \frac{1}{a} \int_{-\pi/a}^{\pi/a} đk \, (K_1(1 - \cos ka) + K_2(1 - \cos 2ka) + \cdots) \, |u(k)|^2.$$

- We thus find the frequency of the normal modes to be

$$\omega(k) \propto \sqrt{K_1(1 - \cos ka) + K_2(1 - \cos 2ka) + \cdots}.$$

In the limit $k \to 0$, the dispersion relation becomes $\omega(k) = v|k|$. While further neighbor interactions change the value of v, they don't change this fundamental feature.

- At low temperatures, only this part of the excitation spectrum is important, so

$$E(T) \approx Na \int_{-\infty}^{\infty} đk \frac{\hbar v|k|}{\exp(\hbar v|k|/kT) - 1} = Na \frac{\pi^2}{6\hbar v}(kT)^2.$$

  Therefore the heat capacity is linear in T. This is a universal property, independent of the detailed interatomic interactions.

We can arrive at this conclusion without ever looking at microscopic details, using a phenomenological (field) approach. Since we only want to look at long-wavelength excitations, we ignore variations in the field on the scale of a, allowing us to replace the displacements u_n with a displacement field $u(x)$. The kinetic energy term becomes $\rho \int dx\, \dot{u}(x)^2/2$, where $\rho = m/a$ is the density. We now use general principles to constrain the form of the potential functional $V[u]$.

- Locality: in most situations, interactions between particles are short-ranged, so we may write

$$V[u] = \int dx\, \Phi(u, du/dx, \ldots)$$

  where Φ is a local potential energy density.

- The above form isn't strong enough, since if we allow infinitely many derivatives, we can construct nonlocal interactions like $u(x)u(x+a)$ by Taylor series. In general, higher derivative terms are 'less local' (in a discrete lattice setup, the $d^n u/dx^n$ takes $n+1$ lattice sites to compute), so locality really implies that higher derivative terms are less significant.

- Translational symmetry: we have $\Phi[u(x) + c] = \Phi[u(x)]$, so Φ cannot depend directly on u.

- Stability: since the fluctuations are about an equilibrium position, there can be no linear term in u or its derivatives. Moreover, the quadratic part of $V[u]$ must be positive definite.

As a result, the most general allowed potential can be expanded as a power series. Each term is of the form $(\partial_x u)^{n_1}(\partial_x^2 u)^{n_2} \ldots$. By locality and translational symmetry, the Fourier transform has 'conservation of momentum', with derivatives becoming factors of k, e.g.

$$\int dx\, (\partial_x u)^2(\partial_x^2 u) \quad \to \quad \int đk_1 đk_2\, k_1 k_2 (k_1 + k_2)^2 u(k_1)u(k_2)u(-k_1 - k_2).$$

In the limit $k \to 0$, higher order derivative terms become unimportant. Moreover, for small deformations we may neglect terms beyond second order in u. Then we are left with only the $(\partial_x u)^2$ term, giving

$$H = \frac{\rho}{2} \int dx\, \left[(\partial_t u)^2 + v^2(\partial_x u)^2\right].$$

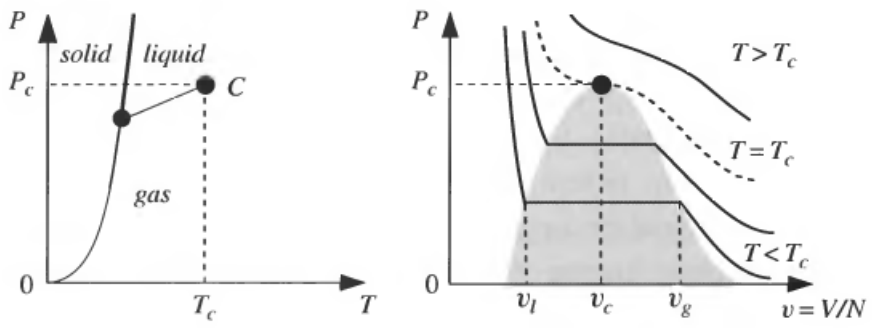
This is the Hamiltonian of a one-dimensional elastic string; its equation of motion is the wave equation. Thus $\omega = v|k|$ for small k, as seen earlier.

Note. In higher dimensions, the result is slightly more complicated. In Fourier space, rotational invariance now allows dependence on k^2, u^2, and $\mathbf{k} \cdot \mathbf{u}$. The final quantity allows us to distinguish longitudinal modes and transverse modes, which may have different velocities. In d dimensions, we find the universal result $E(T) \propto T^{d+1}$, giving a heat capacity proportional to T^d.

1.2 Phase Transitions

Phase transitions cause sudden changes in thermodynamic response functions. All such response functions can be derived from the free energy, which implies that the free energy has a singularity. But since the free energy is analytic for any finite system (since the partition function is), we conclude that phase transitions can only occur in the thermodynamic limit $N \to \infty$.

Example. The liquid-gas phase transition is shown below.



We observe that:

- In the (T, P) plane, the phase transition occurs along a line that terminates at the critical point (T_c, P_c).
- In the (v, P) plane (where $v = V/N$), the transition appears as a coexistence interval, corresponding to a mixture of gas and liquid of densities $\rho_g = 1/v_g$ and $\rho_l = 1/v_l$. (Starting from the van der Waals equation, this interval is found using the Maxwell construction $\int d\mu = 0$.)
- It is possible to go from the gas phase to the liquid phase continuously (without passing through a phase transition), so there is no fundamental difference between them; they are the same phase!

Mathematically, the free energy of the system is analytic, except for a branch cut along the phase boundary. Moreover, near the critical point, we find that:

- As $T \to T_c^-$, ρ_l and ρ_g both converge to ρ_c.
- As $T \to T_c^+$, the pressure vs. volume isotherms become more and more flat, so the isothermal compressibility $\kappa_T = -(\partial V/\partial P)|_T/V$ diverges.
- The fluid appears 'milky' close to criticality (critical opalescence). This shows that collective fluctuations are occurring at wavelengths long enough to scatter visible light.

Indeed, we will see that long-range correlations are characteristic of criticality.

Example. A detailed analysis of the van der Waals gas. The equation of state is

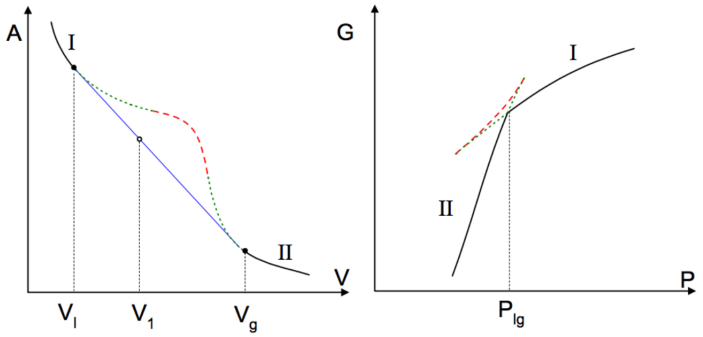
$$(p + a/V^2)(V - b) = NkT.$$

To calculate the free energy, note that $p = (-\partial F/\partial V)_T$, so

$$p = \frac{NkT}{V - b} - \frac{a}{V^2}, \quad F = F_{\text{ideal}} - NkT \log(1 - b/V) - a/V.$$

In general, an equation of state is insufficient to recover the free energy (e.g. the monatomic and diatomic gases have the same equation of state); here, we fix the integration constant, which is an arbitrary function of T, by demanding the van der Waals gas reduce to the ideal gas at high V.

These additional terms mean that there are certain unstable states. The simplest way to see this is to note that $\partial p/\partial V|_T$ can be positive, so the gas is mechanically unstable. More abstractly, we can have $\partial^2 F/\partial V^2$ negative, so the free energy is not convex. The Gibbs free energy, which is its Legendre transform, will be multivalued, but only the state with lowest Gibbs free energy is thermodynamically stable. The unstable states are supercooled gas or superheated liquid.



As shown in the figure above, the physical states can be found by taking the convex hull of the free energy; the physical states are mixtures of liquid and gas. All of these states are mapped to the same "kink" of the Gibbs free energy. This also serves as an example where nonanalyticity can appear from thermodynamic conditions.

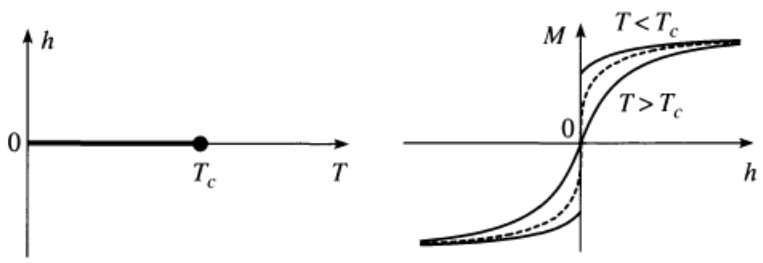
To identify the location of the kink, we note that

$$\oint \frac{\partial G}{\partial p}\, dP = 0.$$

Since $\partial G/\partial p = V$, the physical states on a pV diagram can be identified by the Maxwell equal area construction. Another way of stating this is that, since $G = N\mu$, we simply set the chemical potentials of the liquid and gas components equal.

Note. If a supercooled gas is unstable, why can it exist at all? Naively, the result $\partial p/\partial V < 0$ means that such a gas should immediately condense into a liquid. However, condensation must begin with small droplets, which turn out to be energetically unfavorable due to the surface tension. Roughly put, the benefit of being condensed scales with volume (bonds in the bulk) but the surface tension cost scales with surface area, because this counts the number of 'missing' bulk bonds. Thus droplets below a critical radius are unstable. Clouds form when water vapor condenses on small dust particles instead.

Example. The ferromagnetic-paramagnetic phase transition. The thermodynamic variables are now the external field h, the magnetization M, and the temperature T, where (h, M) is analogous to (P, v). The phase diagram is closely analogous to the liquid-gas phase transition, but it is simpler due to the symmetry $h \to -h$, which ensures that the critical point occurs at $h_c = 0$.



To flesh out the analogy, the liquid and gas phases are analogous to the 'spin up' and 'spin down' phases (except that the order parameter is density, rather than magnetization). The 'paramagnetic' phase that occurs for $T > T_c$ is analogous to the supercritical fluid. For $T < T_c$, coexistence between the spin up and spin down phases can occur, as there are multiple allowed M values for $h = 0$ and fixed T. This is realized by having large magnetic domains that are one or the other. As in the liquid-gas case, we see a divergence in the response function $\partial M/\partial h$ as we approach the critical point from above.

Example. More complicated order parameters. In an Ising antiferromagnet on a simple cubic lattice, the order parameter would be the staggered magnetization

$$\sum_n (-1)^n \langle s_n \rangle$$

where n alternates even/odd between adjacent sites. In a nematic liquid crystal, the order parameter is the orientation of the long molecules, which is a vector mod $\{\pm 1\}$. In the liquid-solid phase transition, which breaks translational symmetry, the order parameter is a Fourier component of the change in density $\delta\rho_{\mathbf{q}}$.

There is no general way to construct an order parameter for a phase transition. Often the experimental discovery of a phase transition comes first; one must then use physical insight to find an order parameter for it.

1.3 Critical Behavior

The singular behavior near a critical point can be characterized by a set of critical exponents, which describe the non-analyticity of various thermodynamic functions. We use the magnet example above for concreteness, defining the reduced temperature $t = (T_c - T)/T_c$.

- Order parameter: the order parameter is an intensive thermodynamic variable that distinguishes phases. In the case of the magnet, it is

  $$m(T, h) = \frac{1}{V} \lim_{V \to \infty} M(T, h).$$

  In the case of zero field, we have

  $$m(T, h = 0) \propto \begin{cases} 0 & T > T_c \\ |t|^\beta & T < T_c \end{cases}.$$

  As we pass the critical temperature, a net magnetization suddenly appears.

- We may also define a critical exponent based on the response to an external field,

$$m(T = T_c, h) \propto h^{1/\delta}.$$

  As we approach the critical point from above, the magnet becomes more and more sensitive to external fields, as it is on a 'tipping point' between magnetization up and down.

- Response function: a related quantity is the response to an infinitesimal external field as a function of temperature, i.e. the susceptibility χ. We typically find

$$\chi_\pm(T, h = 0) \propto |t|^{-\gamma_\pm}$$

  where $\gamma > 0$, so the susceptibility diverges. Often, symmetry forces $\gamma_+ = \gamma_-$.

- Heat capacity: the heat capacity is simply the thermal response function. It behaves as

$$C_\pm(T, h = 0) \propto |t|^{-\alpha_\pm}.$$

  In general, α can be positive (divergence) or negative (finite, possibly with a cusp). In some cases, we can have $\alpha = 0$.

Next, we show that the divergence of the susceptibility directly implies the existence of long-range correlations.

- We begin with the (Gibbs) partition function in a magnetic field h, given by

$$Z(h) = \operatorname{tr} \exp(-\beta H_0 + \beta h M)$$

  where H_0 is the internal energy of the magnet, and the hM term converts this to the Gibbs free energy. The trace is a sum over microscopic degrees of freedom, classical or quantum.

- The equilibrium magnetization can be found by differentiating Z,

$$\langle M \rangle = \frac{\partial \log Z}{\partial(\beta h)} = \frac{1}{Z} \operatorname{tr} \left(M \exp(-\beta H_0 + \beta h M) \right)$$

  The susceptibility is defined as

$$\chi = \frac{\partial \langle M \rangle}{\partial h} = \beta \left(\langle M^2 \rangle - \langle M \rangle^2 \right)$$

  where the second step follows by direct calculation. More generally, it works because $\log Z$ is a cumulant generating function for the magnetization.

- The overall magnetization may be computed from the local magnetization as

$$M = \int d\mathbf{r}\, m(\mathbf{r}).$$

  Then we find that

$$\chi = \beta \int d\mathbf{r} d\mathbf{r}' \, \langle m(\mathbf{r}) m(\mathbf{r}') \rangle - \langle m(\mathbf{r}) \rangle \langle m(\mathbf{r}') \rangle.$$

- Applying translational symmetry, we may simplify this to

$$\chi = \beta \int d\mathbf{r} d\mathbf{r}' \, \langle m(\mathbf{r} - \mathbf{r}')m(\mathbf{0})\rangle - \langle m\rangle^2 = \beta V \int d\mathbf{r} \, \langle m(\mathbf{r})m(\mathbf{0})\rangle - \langle m\rangle^2$$

  where integrating over the CM coordinate gave a factor of volume.

- The integrand is simply the connected correlation function

$$\langle m(\mathbf{r})m(\mathbf{0})\rangle_c = \langle (m(\mathbf{r}) - \langle m(\mathbf{r})\rangle)(m(\mathbf{0}) - \langle m(\mathbf{0})\rangle)\rangle$$

  which determines the correlation of fluctuations away from the equilibrium magnetization. Hence the susceptibility measures the statistical coherence of the system. Typically, the correlation decays exponentially with scale ξ, the correlation length.

- Since the largest possible value of the correlation function is bounded, the fact that $\chi \to \infty$ at the critical point implies that $\xi \to \infty$, specifically $\xi_\pm(T, h = 0) \propto |t|^{-\nu_\pm}$, giving long ranged correlations as in critical opalescence. This motivates the introduction of statistical fields.

Example. The Ising model in mean field theory. Defining N spins $s_i = \pm 1$ on a lattice,

$$E = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j.$$

Defining the average magnetization $m = \sum_i s_i/N$, the effective free energy per spin is defined by

$$Z \sim \int dm \, e^{-\beta N f(m)}$$

and the saddle point approximation states that the equilibrium magnetization $\langle m\rangle$ is the magnetization that minimizes $f(m)$. It is difficult to calculate $f(m)$. In mean field theory, we approximate

$$E(m) \approx N \left(-Bm - \frac{1}{2} Jqm^2 \right)$$

where q is the number of neighbors of each spin. Intuitively, this is what we would get if the spins were distributed randomly, and also what we would get if they could all interact with each other. In both cases, we're ignoring the spatial structure of the problem, in particular how "clumps" of flipped spins may form; instead we assume every spin sees only the mean field.

Assuming that mean field theory is applicable here, we have

$$e^{-\beta N f(m)} \approx \Omega(m) e^{-\beta E(m)}, \quad \Omega(m) = \binom{N}{M}, \quad M = N \frac{m+1}{2}$$

where M is the number of spins up. Applying Stirling's approximation,

$$f(m) \approx -Bm - \frac{1}{2} Jqm^2 - T \left(\log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m) \right).$$

A little bit of algebra shows that $f(m)$ is minimized when

$$m = \tanh(\beta B + \beta J q m).$$

When $B = 0$, we have a continuous phase transition.

Note. The first remarkable fact about critical exponents is that the critical exponents calculated in our mean field Ising model, and with the van der Waals equation, are exactly the same! This is not surprising because they have the same generic form when Taylor expanded to low order; the Landau–Ginzburg Hamiltonian we define below exemplifies this form. The second remarkable fact is that these results are wrong in $d = 3$, but the right answers are exactly the same! This is universality, a deeper phenomenon we will explain with the RG.

Note. Why can we ignore quantum effects in everything we do below? We can do this when the correlation length ξ is much greater than the de Broglie wavelength λ, and for a system with characteristic speed v, $\lambda = v\hbar/k_BT$. Hence quantum effects are negligible at high temperatures or at critical points. The only exception is critical points at zero temperature, which may correspond to quantum phase transitions.

2 Landau Theory

2.1 Landau–Ginzburg Hamiltonian

Near the critical point, the divergence of the correlation length implies that long-range excitations become important; hence we may coarse grain and describe the system with a statistical field.

- For example, for magnetic systems, the relevant excitations are collective excitations of spins. The fact we are coarse-graining translates into the requirement that $\mathbf{m}(\mathbf{x})$ has no Fourier components near the UV cutoff $\Lambda \sim 1/a$.
- We have defined $\mathbf{m}(\mathbf{x})$ as a map $\mathbb{R}^d \to \mathbb{R}^n$, but this formalism may be used to describe a variety of physical situations:
  - The case $n = 1$ describes liquid-gas transitions, binary mixtures, and uniaxial magnets.
  - The case $n = 2$ describes Bose–Einstein condensates, superfluidity, superconductivity, and planar magnets. In the former cases, the free energy gives the Gross–Pitaevskii equation of motion, and the order parameter is the 'macroscopic wavefunction'.
  - The case $n = 3$ describes classical magnets.
- Let H_{mic} be the microscopic Hamiltonian. Then it is possible to write the microscopic partition function $Z = \operatorname{tr} \exp(-\beta H_{\text{mic}})$ as

$$Z = \int \mathcal{D}\mathbf{m}(\mathbf{x})\, W[\mathbf{m}(\mathbf{x})]$$

  where $\mathbf{m}(\mathbf{x})$ is a coarse-grained field and the measure $D\mathbf{m}(\mathbf{x})$ includes a factor for integrating out higher-frequency modes.
- This yields the coarse-grained Hamiltonian

$$\beta H[\mathbf{m}(\mathbf{x})] = -\log W[\mathbf{m}(\mathbf{x})].$$

  Computing the weighting factor W is analytically intractable. However, we may apply general principles to narrow down the possible form of the coarse-grained Hamiltonian H.
- Locality and translational invariance imply that we may write

$$\beta H = \int d\mathbf{x}\, \Phi[\mathbf{m}(\mathbf{x})].$$

- We take terms with at most a few derivatives. This is partly motivated by locality, because an infinite number of derivatives allows nonlocality. Alternatively, we know that higher derivative terms are suppressed in slowly varying field configurations, which are the ones we focus on. This is really a primitive RG argument which we can justify formally later.
- We assert that Φ is analytic in $\mathbf{m}$ and its derivatives, and hence can be expanded in a Taylor series. Note that while if we had nonanalytic behavior at the microscopic scale (e.g. for a potential $V \sim |x|$ we have a Boltzmann weight $e^{-|x|}$), coarse-graining washes out these features in a way analogous to the central limit theorem.

- Since there is rotational symmetry at the microscopic level, it must be preserved at the mesoscopic level. Then we may only have rotationally invariant terms, e.g.

$$m^2 = m_i m_i, \quad (\nabla \mathbf{m})^2 = \partial_\alpha m_i \partial_\alpha m_i, \quad (\nabla^2 \mathbf{m})^2 = (\partial_\alpha \partial_\alpha m_i)(\partial_\beta \partial_\beta m_i).$$

  The original setup is also $\mathbb{Z}_2$ symmetric under $m \to -m$, so all terms should be even in m.

- Applying these principles gives the Landau–Ginzburg Hamiltonian, originally used to describe superconductivity,

$$\beta H = \beta F_0 + \int d\mathbf{x} \left(\frac{t}{2} m^2 + u m^4 + \frac{K}{2} (\nabla \mathbf{m})^2 + \cdots - \mathbf{h} \cdot \mathbf{m} \right).$$

  Here, we have included an external field term $\mathbf{h} \cdot \mathbf{m}$ term, and absorbed a factor of β into the constants. The overall constant βF_0 usually won't matter; there are higher order terms in m and h in the expansion, but we won't need them to describe phase transitions.

- Stability implies that the probability should not diverge for infinitely large field configurations. In the case of the Hamiltonian above, this requires $u > 0$. The probability should also not diverge for oscillatory field configurations, giving $K > 0$.

- The variables t, u, and K are phenomenological parameters that we cannot compute directly. Unlike the couplings in H_{mic}, they have nontrivial (but analytic) dependence on thermodynamic variables, such as P and T, as a result of the coarse-graining procedure.

Note. Here we'll outline the general plan of attack for the rest of these notes. Generally, a thermodynamic system has fluctuations on all length scales ranging from the lattice spacing to the system size. To get the exact answer, we must sum over all possible fluctuations at all length scales, but this is almost always computationally intractable. To split the problem into pieces, we can imagine first summing over fluctuations at the smallest scales, yielding a new effective Hamiltonian for the remaining degrees of freedom; this defines an RG flow on the space of Hamiltonians. By iterating this procedure, we can "integrate out" all degrees of freedom on microscopic scales, focusing only on the macroscopic physics we can observe. For example, if we sum over all fluctuations, we just end up with the free energy, which depends on only the mean magnetization.

In Landau–Ginzburg theory, we integrate out microscopic but not mesoscopic degrees of freedom, simplifying the problem at the cost of making the coefficients in the Landau–Ginzburg Hamiltonian phenomenological. In mean field theory, we ignore the mesoscopic degrees of freedom as well. Hence mean field theory amounts to ignoring all RG flow on the way from mesoscopic to macroscopic scales. (Since mean field theory is such a crude approximation, there are many ways to reach it; we saw one in the previous section.) Later, we will attempt to account for the fluctuations perturbatively. When this fails, we'll turn to a full RG analysis.

2.2 Mean Field Theory

We can evaluate the partition function by focusing on configurations where the integrand $e^{-\beta H}$ is maximized (the 'saddle point approximation'). The crudest version of this is mean field theory.

- Since $K > 0$, fluctuations in the field are suppressed, so the greatest contribution comes from constant field configurations. Then

$$Z \approx e^{-\beta F_0} \int d\mathbf{m} \exp\left(-V \left(\frac{t}{2} m^2 + u m^4 + \cdots - \mathbf{h} \cdot \mathbf{m} \right) \right).$$

This simpler form is called the Landau free energy, as opposed to the more sophisticated Landau–Ginzburg Hamiltonian. This is a very crude approximation; it essentially says that the effective Hamiltonian is unchanged by integrating out *all* fluctuations. Thus mean field theory should only work when our original Landau–Ginzburg Hamiltonian is RG stable.

- In the limit $V \to \infty$ the integral is dominated by the value of $\mathbf{m}$ that maximizes the integrand,

$$\beta F = -\log Z \approx \beta F_0 + V \min(\Psi(\mathbf{m})), \quad \Psi(\mathbf{m}) = \frac{t}{2}m^2 + um^4 + \cdots - \mathbf{h}\cdot\mathbf{m}.$$

  To minimize the final term, we must have $\mathbf{m} = \overline{m}\hat{h}$. Carrying out the minimization, we have

$$t\overline{m} + 4u\overline{m}^3 + \cdots - h = 0.$$

- We note that while $\Psi(\mathbf{m})$ is analytic, the free energy F may not be, because the minimization procedure is not analytic. As usual, the source of the nonanalyticity is the thermodynamic limit $V \to \infty$, which is what allowed us to only consider the minimum in the first place.

This crude approximation turns out to be sufficient to model some phase transitions.

- For $t > 0$, the cubic term is not necessary for stability and can be ignored. We have $\overline{m} \approx h/t$, giving paramagnetic behavior, since $\overline{m} \to 0$ as $h \to 0$. The susceptibility diverges as $t \to 0$.
- For $t < 0$, the quartic term is required to ensure stability. For small enough h, there are two local minima, with the global minimum being aligned with h.
- In the limit $h \to 0$, the magnetization still remains nonzero, indicating ferromagnetism. The sign of m chosen depends on the system's history, or small fluctuations.
- Given the above considerations, we see there is a line of phase transitions at $h = 0$ and $t < 0$.
- To match this model to our magnet, we require $t(T)$ to monotonically increase in temperature and cross zero at $T = T_c$. The most generic choice that works is

$$t(T) = a(T - T_c) + O((T - T_c)^2).$$

  Now we have just the same thing as the Taylor expansion of our mean field Ising result.

- For stability, we require $K(T) > 0$ near the critical point, so generically

$$K(T) = K + O(T - T_c), \quad K > 0.$$

  We could add more structure, e.g. by imposing symmetries, but this will be enough.

Note. Other choices are not generic. Suppose we expand

$$t(T) = \sum_n a_n (T - T_c)^n.$$

Each of the coefficients a_n is generically a complicated function of temperature. Now, there is a single arbitrary choice on the right-hand side, i.e. the critical temperature T_c, and it is chosen so that there is indeed a critical point at T_c by setting $a_0 = 0$. We would see the same behavior if the first nonzero term was a_3, but this would require tuning $a_0 = a_1 = a_2 = 0$, which isn't possible for a generic function. It is possible for symmetries to force a_3 to be the first nonzero term, but there are no such symmetries here. Conversely, as long as we're working with a generic system, every term not forbidden by symmetries must be present.

Note. We can apply similar reasoning to the restriction $u > 0$. We could still have stability if

$$H \sim -m^2 - m^4 + m^6.$$

However, this is ruled out by the requirement that we have a continuous phase transition as $T \to T_c$. To get continuity, we require the minimum to be at $m = 0$ at the critical point, but this can only happen if the coefficients of m^2 and m^4 go to zero simultaneously, which is not generic. The above Hamiltonian is instead a generic prototype for a discontinuous phase transition.

While Landau theory can provide some insight into discontinuous phase transitions, it doesn't work as well quantitatively because it's based on a Taylor series in m, and m may jump to a large value in a discontinuous phase transition.

We can now use our setup to compute critical exponents.

- Explicitly minimizing the free energy in the ferromagnetic phase, we find magnetization

  $$\overline{m}(h = 0) = \sqrt{\frac{a}{4u}(T_c - T)}$$

  near the critical point, giving a critical exponent of $\beta = 1/2$. We focus on the critical exponent because it is universal, i.e. independent of the material properties, which would influence the values of a and u. As expected, $\overline{m}$ vanishes near the critical point, so it will be justified to expand to lowest nontrivial order in $\overline{m}$ below.

- Along the critical isotherm, the response is

  $$\overline{m}(t = 0) = \left(\frac{h}{4u}\right)^{1/3}$$

  which gives the critical exponent $\delta = 3$.

- Since we have $h = t\overline{m} + 4u\overline{m}^3$, it is easier to calculate the inverse susceptibility,

  $$\chi^{-1} = \left.\frac{\partial h}{\partial \overline{m}}\right|_{h=0} = t + 12u\overline{m}^2 = \begin{cases} t & t > 0 \\ -2t & t < 0. \end{cases}$$

  We see that $\gamma_+ = \gamma_- = 1$, with the susceptibility diverging on both sides of the phase transition. The ratio of amplitudes is also predicted to be universal, with a value of 2.

- The free energy for $h = 0$ is

  $$\beta F = \beta F_0 + V\Psi(\overline{m}) = \beta F_0 + V \begin{cases} 0 & t > 0 \\ -t^2/16u & t < 0 \end{cases}.$$

  We may compute the heat capacity from the free energy as $C = -T\partial^2 F/\partial T^2$. Near the critical point, we have $\partial/\partial T \approx a\partial/\partial t$, giving

  $$C(h = 0) \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(kT_c\beta F) = C_0 - (Vka^2T_c^2/8u)\theta(t).$$

  That is, we find a discontinuity in the heat capacity; the critical exponent is $\alpha = 0$.

Note. The Ehrenfest classification defines a phase transition as n^{th} order if the n^{th} derivative of the free energy is discontinuous. First order phase transitions are thus characterized by jumps in the order parameter, while second order phase transitions are characterized by jumps in the susceptibilities/response functions.

However, in many cases, the susceptibilities diverges rather than jumps, as we saw above. In the modern definition, we classify phase transitions as 'discontinuous' (corresponding to first order) and 'continuous' (which includes everything else).

2.3 Symmetry Breaking

The ferromagnetic phase demonstrated above exhibits spontaneous continuous symmetry breaking for $h = 0$ and $n > 1$. Long-range order is established, so that the state of the magnet no longer has the full symmetry of the microscopic Hamiltonian. However, it remains true that a rotation leaves the energy invariant. As a result, rotations that vary slowly in space can cost an arbitrarily low amount of energy. These low-energy, 'gapless' excitations are called Goldstone modes.

Example. Superfluidity. In the superfluid phase, there is a macroscopic occupation of a single quantum ground state, which can be described by a complex order parameter

$$\psi(\mathbf{x}) = |\psi(\mathbf{x})|e^{i\theta(\mathbf{x})}.$$

More precisely, $\psi(\mathbf{x})$ is the expectation value of the field annihilation operator $\hat{\psi}(\mathbf{x})$. Now, since the phase θ is not observable, the coarse-grained Hamiltonian cannot depend on it, giving

$$\beta H = \beta F_0 + \int d\mathbf{x} \left(\frac{K}{2}|\nabla\psi|^2 + \frac{t}{2}|\psi|^2 + u|\psi|^4 + \cdots \right).$$

This is in fact equivalent to our Landau–Ginzburg Hamiltonian for $n = 2$. In the superfluid phase, the magnitude of $|\psi|$ is fixed to $\overline{\psi}$, but the global phase is not. Excitations with varying phase then have the Hamiltonian

$$\beta H = \beta H_0 + \frac{\overline{K}}{2} \int d\mathbf{x}\, (\nabla\theta)^2$$

where $\overline{K} = K\overline{\psi}^2$, as expected from global phase rotation symmetry. In particular, taking a Fourier transform, we find

$$\beta H = \beta H_0 + \frac{\overline{K}}{2} \sum_{\mathbf{q}} q^2 |\theta(\mathbf{q})|^2$$

which gives a soft mode, since the energy goes to zero in the limit of long wavelengths. Note that this Hamiltonian has no kinetic term, which means there are no dynamics. This is acceptable because we are only concerned with the thermodynamics.

Note. The variable θ is only defined up to multiples of 2π, which makes evaluation of the partition function difficult. While we can argue that this doesn't matter for small fluctuations, there exist excitations which have nontrivial phase winding which have thermodynamic effects. We will see this explicitly for vortices, which cause the Kosterlitz–Thouless phase transition.

Note. Suppose the symmetry group of the free energy is G and the symmetry group of a single ground state is $H \subset G$. Then the space of ground states is the coset space G/H, and the dimension of the coset space is the number of Goldstone bosons; they generate translations in the coset space.

In the case $n = 1$, there is no continuous symmetry linking the two ferromagnetic phases, so we have instead broken a discrete symmetry. If the system has regions with different ferromagnetic phases, they are linked by sharp domain walls.

Example. Domain walls for magnetism. For simplicity, we take $d = 1$. By varying the Landau–Ginzburg Hamiltonian, we find that the energy-minimizing configuration satisfies

$$K\frac{\partial^2 m}{\partial x^2} = tm + 4um^3.$$

The solution to this equation satisfying boundary conditions $m(x \to \pm\infty) = \pm\overline{m}$ is

$$m(x) = \overline{m}\tanh\frac{x - x_0}{w}, \quad w = \sqrt{\frac{2K}{-t}}.$$

The width w of the domain wall diverges as $(T_c - T)^{-1/2}$ at the critical point. The free energy cost of the domain wall is a constant, not dependent on the length L of the space; in general dimension it is proportional to the domain wall area L^{d-1}.

Note. In the modern view, we say that two points in a phase diagram are the 'same phase' if they can be connected by smoothly deforming the Hamiltonian without ever changing its symmetries, and without crossing a phase transition.

For example, in the $n = 1$ Landau–Ginzburg model with zero external field, the phase diagram is one-dimensional, as one can only vary T. The phases above and below T_c are different phases, distinguished by the presence of the $m \to -m$ symmetry. However, if we allowed an external field from the outset, then the $m \to -m$ symmetry wouldn't exist. Accordingly, all generic points (t, h) on the phase diagram can be connected, so they are the same phase. Similarly, liquids and gases are the same phase, as they can be connected by going through a supercritical fluid.

In this new view, there is now a line of discontinuous phase transitions embedded within the single phase. Intuitively, it comes from the merging of two continuous phase transitions. As we pass through the line, we regain the $m \to -m$ symmetry and immediately break it.

If we don't require any symmetries at all, then all the phases we've talked about so far collapse to a single phase, the "topologically trivial" phase! The only phase transitions that survive are topological phase transitions, which we'll see later.

Note. To analyze spontaneous symmetry breaking formally, we turn on a small symmetry-breaking field h for a system of finite size N. Then we take the limit $N \to \infty$ followed by the limit $h \to 0$. If the resulting order parameter depends on how we took the limit $h \to 0$, then we have spontaneous symmetry breaking. The order of limits is essential; if we take $h \to 0$ first we always get a trivial result. Rigorous mathematical results about spontaneous symmetry breaking are proven using this framework.

All continuous phase transitions are associated with spontaneous symmetry breaking. However, the converse is not true. Representation theory can help illuminate when the converse fails to hold.

- As two examples, the liquid-solid phase transition breaks translational symmetry, but in three dimensions, it is a discontinuous phase transition. And the liquid-gas phase transition breaks no symmetries and is discontinuous. Both of these are compatible with the statement above.

- One prototype example is the Hamiltonian

$$H \sim -m^2 - m^4 + m^6$$

  which we've discussed before; as the coefficient of m^2 varies, we have a discontinuous but symmetry breaking phase transition.

- Another prototype can come from cases where the Hamiltonian has a cubic term. For example, in the liquid-solid transition, the total density is $\rho_0 + \delta\rho$. There is no symmetry between the densities $\rho_0 \pm \delta\rho$, so we can have a cubic term $(\delta\rho)^3$ in the Hamiltonian, which generically yields a discontinuous phase transition.

- Generally we assign discontinuous phase transitions less physical meaning than continuous ones, as they can happen "without any particular reason" when one minimum in a large free energy landscape happens to dip below another.

- In general, suppose the order parameter transforms in a representation of a group G. We may break this representation into irreps R_i. Now, each term in the Landau free energy must be invariant under G, so the terms may only be G-invariant homogeneous polynomials in the coefficients. Specifically, there can be n^{th} order terms associated with the coefficients of the irrep R_i exactly if $1 \subset (R_i)^{\otimes n}$ where 1 is the trivial representation.

- As long as we aren't dealing with the trivial representation, all linear terms are forbidden. Usually a quadratic term is allowed, because we mostly use real representations, and $1 \subset R_i \otimes R_i^*$. Thus, the presence of symmetry-breaking discontinuous phase transitions often boils down to the existence of irreps R_i so that $1 \subset R_i^{\otimes 3}$.

Example. Liquid crystals. The local orientation of a liquid crystal is described by a director $\mathbf{n} \sim -\mathbf{n}$. One way to get an order parameter invariant under this transformation is to use a quadratic; the simplest option is

$$Q_{ij} = \langle n_i n_j \rangle - \frac{1}{3}\delta_{ij}$$

where the diagonal term is subtracted so that $Q_{ij} = 0$ in the disordered state. By rotational invariance, the Landau–Ginzburg Hamiltonian must be invariant under $Q \to RQR^T$, and the simplest terms that work have the form $\operatorname{tr}(Q^n)$. The leading order terms are

$$F = \frac{1}{2}(\partial_i Q_{jk})^2 + \alpha_2 \operatorname{tr} Q^2 + \alpha_3 \operatorname{tr} Q^3 + \alpha_4 (\operatorname{tr} Q^2)^2 + \dots$$

where there is no linear term as expected since $\operatorname{tr} Q = 0$. The cubic term tells us the phase transition is generically discontinuous. In terms of representation theory, the $\mathbb{Z}_2$ symmetry forces us to use the 5 of $SO(3)$, and any power of 5 contains the 1.

3 Fluctuations

3.1 Scattering and Fluctuations

In the previous section, we have only considered mean field theory. However, there will generally be fluctuations about the mean field result, which can be detected by scattering experiments.

- Suppose we send a probe of momentum $\mathbf{k}$ into a system in state $|i\rangle$ that is scattered into momentum $\mathbf{k}+\mathbf{q}$, with the system in state $|f\rangle$. The scattering amplitude is

  $$A(\mathbf{q}) \propto \langle \mathbf{k}+\mathbf{q}, f|U|\mathbf{k}, i\rangle \propto \sigma(\mathbf{q})\rho(\mathbf{q})$$

  where U is the scattering potential. Here $A(\mathbf{q})$ picks up the Fourier component $U(\mathbf{q})$, which can be separated into a local form factor $\sigma(\mathbf{q})$ describing scattering from each individual element in the sample, and a global density $\rho(\mathbf{q})$.

- For example, light scattering senses the atomic density, electron scattering measures the charge density, and neutron scattering can be used to measure magnetization density.

- Since the fluctuations are dynamic, we typically only can see time-averaged configurations, so the averaged scattering intensity is

  $$S(\mathbf{q}) \propto \langle |A(\mathbf{q})|^2 \rangle \propto \langle |\rho(\mathbf{q})|^2 \rangle.$$

  We see that we can pick up long-wavelength fluctuations by looking at nearly forward scattering, or by sending in small $\mathbf{k}$.

Given the above discussion, we now turn to computing $\langle |\rho(\mathbf{q})|^2 \rangle$ for the Landau–Ginzburg Hamiltonian, in which case the 'density' is the magnetization density.

- From the previous section, we know that the probability of a field configuration is

  $$P(\mathbf{m}(\mathbf{x})) \propto \exp\left[-\int d\mathbf{x} \left(\frac{K}{2}(\nabla m)^2 + \frac{t}{2}m^2 + um^4 \right) \right].$$

- Let the mean field configuration be $\mathbf{m}(\mathbf{x}) = \overline{m}\hat{e}_1$. Small fluctuations can be parametrized as

  $$\mathbf{m}(\mathbf{x}) = (\overline{m} + \phi_l(\mathbf{x}))\hat{e}_a + \sum_{\alpha=2}^{n} \phi_{t,\alpha}\hat{e}_\alpha.$$

- Plugging into the Landau–Ginzburg Hamiltonian and expanding to quadratic order in the fluctuations, we have

  $$\beta\Delta H = \frac{K}{2}\int d\mathbf{x}\, (\nabla\phi_l)^2 + \frac{\phi_l^2}{\xi_l^2} + (\nabla\phi_t)^2 + \frac{\phi_t^2}{\xi_t^2}$$

  where the characteristic lengths are

  $$\frac{K}{\xi_l^2} = t + 12u\overline{m}^2 = \begin{cases} t & t > 0 \\ -2t & t < 0 \end{cases}, \quad \frac{K}{\xi_t^2} = t + 4u\overline{m}^2 = \begin{cases} t & t > 0 \\ 0 & t < 0. \end{cases}$$

  As expected, transverse and longitudinal fluctuations behave the same in the unordered phase, while transverse fluctuations have no restoring force in the ordered phase since they are Goldstone modes.

- Taking the Fourier transform, the probability reduces to

$$P(\phi) \propto \prod_{\mathbf{q}} \exp\left[-\frac{K}{2}(q^2+\xi_l^{-2})|\phi_{l,\mathbf{q}}|^2\right] \exp\left[-\frac{K}{2}(q^2+\xi_t^{-2})|\phi_{t,\mathbf{q}}|^2\right]$$

  where we used the property $\phi_{\mathbf{q}} = \phi^*_{-\mathbf{q}}$, since $\phi(\mathbf{x})$ is real. Therefore we see that fluctuations in momentum space are uncorrelated Gaussians.

Note. We need to be a bit careful with counting degrees of freedom to arrive at the above conclusion. For simplicity, we consider the one-dimensional case,

$$P(\phi) \sim \prod_q \exp\left[\frac{f(q)}{2}|\phi_q|^2\right]$$

We know that there is one real degree of freedom for each $\mathbf{x}$ value, but there are formally two for each $\mathbf{q}$ value; this is because of the constraint $\phi_q = \phi^*_{-q}$. To see this explicitly, we can perform the product for q and $-q$ pairs, assuming $f(q) = f(-q)$, to get

$$P(\phi) \sim \prod_{q>0} \exp\left[f(q)\left((\operatorname{Re}\phi_q)^2 + (\operatorname{Im}\phi_q)^2\right)\right]$$

which has two degrees of freedom per q value, but only half as many q values. Now, the quantities $(\operatorname{Re}\phi_q)^2$ and $(\operatorname{Im}\phi_q)^2$ are all independent Gaussians, so they have correlation functions

$$\langle(\operatorname{Re}\phi_q)^2\rangle = \langle(\operatorname{Im}\phi_q)^2\rangle = \frac{1}{2f(q)}$$

with zero cross-correlations. Then the original complex degrees of freedom have correlation

$$\langle\phi_q\phi_{-q}\rangle = \frac{1}{f(q)}$$

with zero cross-correlations; in particular, note that $\langle\phi_q^2\rangle = 0$. The upshot is that we can either integrate over all q treating the ϕ_q as real degrees of freedom, or integrate over only $q > 0$ (in higher dimensions, $q_1 > 0$) treating the ϕ_q as complex degrees of freedom.

- Applying the above result, we find that

$$\langle\phi_{\alpha,\mathbf{q}}\phi_{\beta,\mathbf{q}'}\rangle = \frac{\delta_{\alpha,\beta}\delta_{\mathbf{q},-\mathbf{q}'}}{K(q^2+\xi_\alpha^{-2})}.$$

- The above calculation predicts that the scattering intensity has the Lorentzian form $S(\mathbf{q}) \propto 1/(q^2+\xi^{-2})$. This is indeed confirmed by neutron scattering experiments, which can pick out the transverse and longitudinal parts by the spin polarization of the neutrons.

3.2 Position Space Fluctuations

To see how the fluctuations behave in position space, we Fourier transform again. In particular, we care about the connected correlation function G^c, which is the correlation function of the fluctuations of m about the mean value $\overline{m}$,

$$G^c_{\alpha,\beta}(\mathbf{x},\mathbf{x}') = \langle\phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x}')\rangle = \frac{1}{V}\sum_{\mathbf{q},\mathbf{q}'} e^{i\mathbf{q}\mathbf{x}+i\mathbf{q}'\mathbf{x}'}\langle\phi_{\alpha,\mathbf{q}}\phi_{\beta,\mathbf{q}'}\rangle.$$

- Plugging in our previous result, we find

$$G^c_{\alpha\beta}(\mathbf{x},\mathbf{x}') = -\frac{\delta_{\alpha,\beta}}{K} I_d(\mathbf{x}-\mathbf{x}'), \quad I_d(\mathbf{x}) = -\int đ\mathbf{q}\, \frac{e^{i\mathbf{q}\mathbf{x}}}{q^2+\xi^{-2}}.$$

  In one dimension, this is a simple contour integral. In three dimensions, it is the integral for the Yukawa potential in field theory. In that case, the angular integration is simple since the angular measure provides the required $\sin\theta$ factor, and the radial integral can be performed by contour integration, giving

$$-I_d(r) = e^{-r/\xi} \times \begin{cases} \xi/2 & d=1, \\ 1/4\pi r & d=3. \end{cases}$$

  However, we will be interested in general dimensions, where the angular integral is harder.

- A more general method is to note that

$$\nabla^2 I_d(\mathbf{x}) = \int đ\mathbf{q}\, \frac{q^2 e^{i\mathbf{q}\mathbf{x}}}{q^2+\xi^{-2}} = \delta^d(\mathbf{x}) + \frac{I_d(r)}{\xi^2}$$

  where $r = |\mathbf{x}|$, by spherical symmetry, giving a differential equation for $I_d(r)$. For $\xi \to \infty$ the equation is Poisson's equation for a point charge at the origin, so we find the Coulomb potential,

$$V(\mathbf{q}) \sim \frac{1}{q^2}, \quad V(r) \sim \frac{1}{r^{d-2}}.$$

  Note that this assumes the potential vanishes at infinity. We cannot do this in $d \le 2$, so in $d=1$ we should add a constant. Also note that the r^0 in $d=2$ should be interpreted as $\log r$, again with an arbitrary constant.

- In general, the differential equation is

$$\frac{d^2 I_d}{dr^2} + \frac{d-1}{r}\frac{dI_d}{dr} = \frac{I_d}{\xi^2} + \delta(r).$$

  The solution is a Bessel function, and for low and high x we have

$$I_d(r) \sim \begin{cases} 1/r^{d-2} & r \ll \xi, \\ e^{-r/\xi}/r^{(d-1)/2} & r \gg \xi. \end{cases}$$

  In the case $r \ll \xi$, the fluctuations don't feel the value of ξ, so we just get back the Coulomb potential. This is a power law, i.e. the system looks 'effectively critical' at scales smaller than ξ. For $x \gg \xi$, we get exponential decay.

- This result is called the Ornstein–Zernicke correlation. We can derive it by using an ansatz $e^{-r/\xi}/r^p$. Alternatively, note that

$$\frac{1}{q^2+\xi^{-2}} = \int_0^\infty dt\, e^{-t(q^2+\xi^{-2})}.$$

  Plugging this into the original integral expression for $I_d(\mathbf{x})$, yields a Gaussian integral over $\mathbf{q}$. Performing it leaves the dt integral over $e^{-S(t)}$ where $S(t)$ is a complicated function, and this integral can be performed by saddle point in the regimes $r \ll \xi$ and $r \gg \xi$.

- From our above work, we see ξ can be interpreted as a correlation length. Focusing on the longitudinal correlators, we define
$$\xi_{\ell,\pm} = B_\pm t^{-\nu_\pm}$$
above and below the critical point, in which case we find
$$\frac{B_+}{B_-} = \sqrt{2}, \quad \nu_\pm = \frac{1}{2}.$$

- We may also calculate bulk susceptibilities by integrating the correlation function, as demonstrated earlier. Near the critical point, most of the contribution comes from $r \ll \xi$, giving
$$\chi_l \propto \int d\mathbf{x}\, G_l^c(\mathbf{x}) \sim \int_0^{\xi_l} \frac{r^{d-1}dr}{r^{d-2}} \propto \xi_l^2$$
which recovers the critical exponents $\gamma_\pm = 1$ and the amplitude ratio 2 found before.

- Experimentally, we find that the correlation actually decays as
$$G(r) \sim \frac{1}{r^{d-2+\eta}}$$
where η is a new critical exponent that our basic theory predicts to be zero. Then we instead have $\chi_l \sim \xi_l^{2-\eta}$.

- The transverse fluctuations instead have long-range (power law) correlations for all $T \leq T_c$, so the transverse susceptibility integral is cut off by the size of the sample,
$$\chi_l \sim \int_0^L \frac{d\mathbf{x}}{x^{d-2}} \propto L^2.$$

We now investigate the Goldstone modes more closely. For simplicity, we switch to the case of superfluid helium, which only has the transverse fluctuations, parametrized by θ.

- As argued above, $I_d(\mathbf{x}) \propto 1/x^{d-2}$ in this case, and we can find the constant of proportionality from Gauss's theorem, giving
$$I_d(x) = \frac{x^{2-d}}{(2-d)\Omega_d} + c_0$$
where c_0 is a constant of integration and Ω_d is the area of a unit sphere in $\mathbb{R}^d$, where for reference
$$\Omega_2 = 2\pi, \quad \Omega_3 = 4\pi, \quad \Omega_4 = 2\pi^2, \quad \Omega_5 = \frac{8}{3}\pi^2.$$
Note that $I_d(x)$ asymptotes to a constant for $d > 2$, but grows without bound for $d \leq 2$.

- To eliminate the constant of integration, note that
$$\langle[\theta(\mathbf{x}) - \theta(0)]^2\rangle = 2\langle\theta^2\rangle - 2\langle\theta(\mathbf{x})\theta(0)\rangle \sim x^{2-d} + \text{const}$$
where the constant comes from both c_0 and $\langle\theta^2\rangle$. The left-hand side is zero for $x = 0$, but we also know that our field description breaks down for $x \sim a$ where a is on the order of the lattice spacing, so it should reach zero for $x \sim a$. We conclude that both $\langle\theta^2\rangle$ and c_0 depend on the lattice spacing, and combine to give
$$\langle[\theta(\mathbf{x}) - \theta(0)]^2\rangle \propto x^{2-d} - a^{2-d}.$$

- To see how this affects order, we take the correlation function of the order parameter,

$$\langle\psi(\mathbf{x})\psi^*(0)\rangle = \overline{\psi}^2\langle e^{i[\theta(\mathbf{x})-\theta(0)]}\rangle = \overline{\psi}^2\exp\left(-\frac{1}{2}\langle[\theta(\mathbf{x})-\theta(0)]^2\rangle\right)$$

  where we used the identity $\langle\exp(\alpha\theta)\rangle = \exp(\alpha^2\langle\theta^2\rangle/2)$, valid for any collection of Gaussian distributed variables.

- Now consider the limit $x \to \infty$. For $d > 2$, the correlation function asymptotes to a constant that is slightly less than $\overline{\psi}^2$, so fluctuations diminish long-range order. For $d \leq 2$, the correlation function asymptotes to zero, showing that long-range order is destroyed!

- This exhibits the Mermin–Wagner theorem: there can be no spontaneous breaking of a continuous symmetry in $d \leq d_l$, where $d_l = 2$ is the lower critical dimension.

- The Mermin–Wagner theorem has a number of caveats and exceptions.
  - Long-range interactions generally lower the critical dimension.
  - The borderline dimension $d_l = 2$ must be treated carefully. There is in fact a phase transition for a 2D superfluid, though there is no long-range order.
  - If a discrete symmetry is broken instead, there are no Goldstone bosons, and the lower critical dimension turns out to be $d_l = 1$. For example, the 2D Ising model has a phase transition, while the 1D Ising model doesn't.

Note. To see this in detail, note that in the 1D Ising model, the energy cost of a domain wall is constant, but the entropy gain increases with the system size. Then in the thermodynamic limit, at any finite temperature, domain walls must form, making the average magnetization zero.

3.3 Saddle Point Fluctuations

We may use a refinement to the saddle point approximation to find how fluctuations modify the free energy, and other macroscopic properties.

- Expanding to quadratic order in the fluctuations, we have

$$Z \approx \exp\left[-V\left(\frac{t}{2}\overline{m}^2 + u\overline{m}^4\right)\right]\prod_i \int \mathcal{D}\phi_i(\mathbf{x})\exp\left[-\frac{K}{2}\int d^d\mathbf{x}\,(\nabla\phi_i)^2 + \frac{\phi_i^2}{\xi_i^2}\right].$$

- Each of the Gaussian kernels is diagonalized by the Fourier transform, with the $\mathbf{q}$ component having the eigenvalue $K(\mathbf{q}) = K(q^2 + \xi^{-2})$. Therefore

$$\log\det K = \sum_{\mathbf{q}}\log K(\mathbf{q}) = V\int đ\mathbf{q}\,\log K(q^2+\xi^{-2})$$

  where the overall Gaussian integral is proportional to $1/\sqrt{\det K}$. To get the factors of $1/2$ right, we need to think of each degree of freedom $\phi_{\mathbf{q}}$ as real, as discussed above.

- Switching to the free energy $f = -(\log Z)/V$, each polarization contributes

$$\Delta f_i = \frac{1}{2}\int đ\mathbf{q}\,\log K(q^2+\xi^{-2}).$$

- The integral above is divergent; it must be cutoff by the lattice spacing $\Lambda \sim 1/a$. This yields a singular contribution on the form a^{-d}. To extract how the integral depends on ξ, note that it can be made dimensionless by scaling q by ξ, picking up a Jacobian factor ξ^{-d}. Therefore

$$\Delta f_i \sim a^{-d} + \xi^{-d}.$$

- Adding a constant to the free energy doesn't matter, so we focus on the latter term. Near the critical point, the singular contribution to the free energy due to the mean field goes as t^2, while the contribution from fluctuations goes as $\xi^{-d} \propto t^{d/2}$. Therefore, for $d < 4$, the fluctuation contribution is dominant!

- We can repeat the logic for the heat capacity $C \propto -\partial^2 f/\partial t^2$, which gives

$$\Delta C_i \sim a^{4-d} + \xi^{4-d}.$$

  The first term modifies the discontinuity in the free energy by a (large) constant, while the second modifies it by $t^{(d-4)/2}$. The mean field theory analysis predicts a critical exponent $\alpha = 0$, giving a simple discontinuity. Then when $d > 4$, the mean field theory result is just shifted by fluctuations, while when $d < 4$ the critical exponent is changed.

- We call $d_u = 4$ the upper critical dimension. For $d > d_u$, the critical exponents of mean field theory are quantitatively correct. Below d_u, our above calculation doesn't work, as it is simply the first-order correction; the higher-order corrections would be even larger. For $d = d_u$ we must be more careful; we'll see below the predictions of mean field theory are correct.

- Note that zero exponents should be interpreted as logarithmic divergences, i.e. $a^0 = \log a$ and $\xi^0 = \log \xi$. Generally logarithms appear whenever zero exponents do.

Note. Review of critical exponents. In MFT, $\beta = 1/2$, $\delta = 3$, $\gamma_\pm = 1$, $\alpha_\pm = 0$, $\nu_\pm = 1/2$, $\eta = 0$.

- Below the critical point, $m(t) \sim |t|^\beta$.

- Along the critical isotherm, $m(t = 0, h) \sim h^{1/\delta}$. This is the only critical exponent involving nonzero h.

- The zero field susceptibility is $\partial m/\partial h \sim |t|^{-\gamma_\pm}$.

- The singularity in the heat capacity is $C_\pm(t) \sim |t|^{-\alpha_\pm}$.

- The correlation length diverges as $\xi_\pm(t) \sim |t|^{-\nu_\pm}$.

- The (longitudinal) susceptibility and correlation length are related by $\chi \sim \xi^{2-\eta}$. Equivalently, at criticality, the correlation function decays as $1/x^{d-2+\eta}$.

To remember the signs, note that everything is defined so the critical exponents are usually positive.

3.4 ∗ Path Integral Methods

We can redo some of the calculations above using path integral methods and more explicit notation.

- We define Fourier transforms by

$$\phi_{\mathbf{k}} = \int d\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\phi(\mathbf{x})$$

where the real field $\phi(\mathbf{x})$ satisfies $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$. The inverse Fourier transform is

$$\phi(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}\phi_{\mathbf{k}} = \int đ\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}.$$

- The path integral $\mathcal{D}\phi(\mathbf{x})$ translates to

$$\mathcal{D}\phi(\mathbf{x}) = \prod_k \int d\phi_{\mathbf{k}} d\phi_{\mathbf{k}}^*$$

when $\phi(\mathbf{x})$ is a complex field, where we consider z and z^* to be independent as usual. In the case of a real field, half of the degrees of freedom on the right are redundant, since $\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$. In this case the integrals can be done by removing the $k_i < 0$ terms for some i. Alternatively, if we naively consider each $\phi_{\mathbf{k}}$ as a single real degree of freedom we get the same answer.

- The basic Gaussian integral we use is

$$\int dx\, e^{-x^2/2a} = \sqrt{2\pi a}.$$

We are only expanding to quadratic order in the fields, so the free energy is

$$F[\phi(\mathbf{x})] = \frac{1}{2}\int d\mathbf{x}\,(\gamma(\nabla\phi)^2 + \mu^2\phi^2), \quad F[\phi_{\mathbf{k}}] = \frac{1}{2}\int đ\mathbf{k}\,(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2.$$

Integrating using the path integral measure gives

$$Z = \int \mathcal{D}\phi(\mathbf{x})\, e^{-\beta F[\phi(\mathbf{x})]} = \prod_{\mathbf{k}} \sqrt{\frac{2\pi T}{\gamma k^2 + \mu^2}}$$

where we keep the temperature explicit. We have a square root since the field is real.

- The free energy density is

$$f = -\frac{T}{V}\log Z = \frac{T}{2}\int đ\mathbf{k}\, \log\frac{2\pi T}{\gamma k^2 + \mu^2}.$$

Next, we translate this to a heat capacity. Using thermodynamics, we have

$$C = -T\frac{\partial^2 F}{\partial T^2}\bigg|_V$$

while using statistical mechanics we have

$$\langle E\rangle = -\frac{\partial \log Z}{\partial\beta}, \quad C = \frac{\partial\langle E\rangle}{\partial T} = \beta^2\frac{\partial^2}{\partial\beta^2}\log Z.$$

One can show these are equivalent, but since critical exponents always come from leading singularities, all that either of these equations really say is $C \sim \partial^2 f/\partial T^2$. Factors of T don't matter, since they are regular, so derivatives $\partial/\partial\beta$ and $\partial/\partial T$ are equivalent. All derivatives can be taken to act on f directly, as the most singular terms in $\partial^n(Tf)$ are of the form $T\partial^n f$.

- Using these simplifications, throwing away constants and less singular pieces, and using the known temperature-dependence of μ^2 right above the critical point,

$$C \sim \frac{\partial^2 f}{\partial T^2} \sim \frac{\partial^2}{\partial T^2} \int đ\mathbf{k} \, \log(\gamma k^2 + (T - T_c)) \sim \int \frac{đ\mathbf{k}}{(k^2 + (T - T_c))^2} \sim \Lambda^{d-4} + (T - T_c)^{(d-4)/2}$$

  matching what we found earlier. If we had been more careful, we would have retained an $Nk_B/2$ term, from equipartition.

Previously, we found the correlation function by directly using the probability distribution for the field. Here, we'll use a methods that are more similar to those in quantum field theory. Along the way, we will relate the correlation function to the Green's function.

- In statistical mechanics, we can differentiate the log of the partition function $Z = \sum e^{-\beta E}$ with respect to β to get cumulants of the energy. This procedure essentially works because the energy and inverse temperature appear together in the form $-\beta E$.

- Similarly, if we add an external field $B(\mathbf{x})$, our free energy becomes

$$F[\phi(\mathbf{x})] = \int dx \, \frac{\gamma}{2}(\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2 + B\phi$$

  and since $Z \sim \sum e^{-\beta F}$ the quantities βB and ϕ are conjugate. Then, for example,

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(\mathbf{x})\delta B(\mathbf{y})} = \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle - \langle\phi(\mathbf{x})\rangle\langle\phi(\mathbf{y})\rangle = \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c$$

  which is indeed the joint cumulant of $\phi(\mathbf{x})$ and $\phi(\mathbf{y})$. We will refer to joint cumulants as connected correlation functions. In most cases we'll evaluate both sides at $B = 0$.

- To perform quadratic path integrals with sources, note that for symmetric G,

$$\int d\mathbf{y} \, \exp\left(-\frac{1}{2}\mathbf{y}^T G^{-1}\mathbf{y} + \mathbf{B}\cdot\mathbf{y}\right) = \sqrt{\det 2\pi G}\, e^{\mathbf{B}^T G\mathbf{B}/2}.$$

  To get this result, we first complete the square,

$$-\frac{1}{2}\mathbf{y}^T G^{-1}\mathbf{y} + \mathbf{B}\cdot\mathbf{y} = -\frac{1}{2}(G^{-1}\mathbf{y} + \mathbf{B})^T G(G^{-1}\mathbf{y} + \mathbf{B}) + \frac{1}{2}\mathbf{B}^T G\mathbf{B}$$

  where the second term goes outside the integral. The Gaussian integral can then be performed by shifting $\mathbf{y}$ and diagonalizing G.

- In momentum space, G is diagonal, with $G(k) = 1/(\gamma k^2 + \mu^2)$, so

$$Z[B(\mathbf{x})] = Z[0] \exp\left(\frac{\beta}{2}\int đ\mathbf{k} \frac{|B_\mathbf{k}|^2}{\gamma k^2 + \mu^2}\right) = Z[0] \exp\left(\frac{\beta}{2}\int d\mathbf{x}d\mathbf{y} \, B(\mathbf{x})G(\mathbf{x}-\mathbf{y})B(\mathbf{y})\right)$$

  where we performed an inverse Fourier transform, and

$$G(\mathbf{x}) = \int đ\mathbf{k} \, \frac{e^{-i\mathbf{k}\mathbf{x}}}{\gamma k^2 + \mu^2}, \quad \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c \propto G(\mathbf{y} - \mathbf{x}).$$

  This recovers the result we found earlier.

- Alternatively, we can stay entirely in position space, where we have

$$\mathbf{y}^T G^{-1}\mathbf{y} \to \int d\mathbf{x}d\mathbf{y}\, \phi(\mathbf{x})G^{-1}(\mathbf{x},\mathbf{y})\phi(\mathbf{y}), \quad G^{-1}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})(-\gamma\nabla_y^2 + \mu^2).$$

  Then it is clear that G is a Green's function for the quadratic free energy, a general result. Intuitively, the correlation function measures how $\mathbf{x}$ and $\mathbf{y}$ fluctuate together, and the Green's function measures how a fluctuation at $\mathbf{x}$ affects $\mathbf{y}$, so it is intuitive they are related.

- The same situation appears in quantum field theory, where the two-point correlation function is called the propagator. However, the situation is more complicated because there are many different Green's functions and accordingly many different correlation functions. We focused on the Feynman Green's function, corresponding to the time-ordered correlator.

- These ambiguities do not appear in statistical field theory, which takes place in Euclidean space. The fields in the correlator commute, so ordering doesn't matter, and we can unambiguously choose the Green's function to be the one that goes to zero at infinity. This Euclidean Green's function is the Wick rotation of the Feynman Green's function.

- The ambiguity in correlators also doesn't appear in the path integral formulation of quantum field theory, where the fields commute. This is because the path integral only computes the time-ordered correlator; it is 'secretly' in Euclidean signature.

- Note that all higher connected correlation functions are trivial; this is because we are dealing with a free theory, so no diagrams with more than two vertices can be connected.

- Finally, we can relate the correlation function to the susceptibility. We define

$$\chi(\mathbf{x},\mathbf{y}) = \frac{\partial\langle\phi(\mathbf{x})\rangle}{\partial B(\mathbf{y})}.$$

  But comparing this to our expressions above gives $\chi(x,y) = \langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle_c$, as shown earlier.

Note. The Ginzburg criterion. Consider approaching the critical point from below. To get a measure for how important the fluctuations are, we compute the ratio

$$R = \frac{\int_0^\xi d\mathbf{x}\,\langle\phi(\mathbf{x})\phi(0)\rangle_c}{\int_0^\xi d\mathbf{x}\,\langle\phi(0)\rangle^2} \sim \frac{1}{m_0^2\xi^d}\int_0^\xi dr\,\frac{r^{d-1}}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2}$$

where we cut off the integrals at ξ since the correlation function decays rapidly outside. According to mean field theory, $m_0 \sim t^{1/2}$ and $\xi \sim t^{-1/2}$, giving $R \sim t^{(d-4)/2}$. Then as we've seen before, the fluctuations dominate at the critical point for $d < 4$. Again, $d = 4$ is indeterminate, since we have a logarithmic divergences.

Note that the overall constant (here, m_0^2) can vary between different materials, so that it can be easy to see fluctuation effects in one system and very hard in others. For example, the critical behavior of some superconductors is very well described by mean field theory given typical experimental resolution in t.

Note. The analogy with quantum field theory. In QFT, the partition function is

$$Z = \int \mathcal{D}\phi\, e^{iS/\hbar}$$

so a Wick rotation $t \to it$ takes a statistical field theory in d spatial dimensions to a QFT in d spacetime dimensions. While the physical interpretations of the results are different, the correlation functions take the same form. For example, a second order phase transition corresponds to a QFT with massless particles, as both have power law correlators. The fact that spontaneous symmetry breaking cannot occur in $d = 1$ translates to the absence of symmetry breaking in $0 + 1$-dimensional QFT (i.e. quantum mechanics) since we may superpose the vacua.

4 The Scaling Hypothesis

We have seen how the leading singular behavior of thermodynamic quantities in the vicinity of the critical point is described by a set of critical exponents, and how fluctuations can modify the values of the critical exponents. In this section, we find relationships between the critical exponents that hold despite the presence of fluctuations.

4.1 The Homogeneity Assumption

We return to the case of the Landau–Ginzburg Hamiltonian, which has a coexistence line for $h = 0$ and $t < 0$, and a critical point at $t = h = 0$.

- In the saddle point approximation, the free energy is

$$f(t,h) = \min\left(\frac{t}{2}m^2 + um^4 - hm\right)_m \propto \begin{cases} t^2/u & h = 0, t < 0 \\ h^{4/3}/u^{1/3} & h \neq 0, t = 0. \end{cases}$$

- It can be checked that for general h and t near the critical point satisfying $t \leq 0$, the singular behavior can be described by a homogeneous function in t and h,

$$f(t,h) = |t|^2 g_f(h/|t|^\Delta)$$

  where Δ is called the gap exponent and g_f is analytic. To get the right limiting behaviors for $t \to 0$ and $h \to 0$, we require

$$\lim_{x\to 0} g_f(x) \sim \frac{1}{u}, \quad \lim_{x\to\infty} g_f(x) \sim \frac{x^{4/3}}{u^{1/3}}.$$

  Here, the $\sim$ means that we are only consider the leading term in both cases. Plugging in the previous result, we find $f \sim |t|^{2-4\Delta/3}$ for $t = 0$. Since t must drop out, $\Delta = 3/2$.

- The homogeneity assumption is that, beyond the saddle point approximation, the singular part of the free energy remains homogeneous, with

$$f_{\text{sing}}(t,h) = |t|^{2-\alpha} g_f(h/|t|^\Delta)$$

  where α and Δ depend on the critical point. Note that there are separate functions g_f for $t \leq 0$ and $t \geq 0$.

- Homogeneity is preserved under differentiation. To see this, note that our homogeneity assumption simply says that f_{sing} has terms of the form $h^i|t|^j$, where $i - j/\Delta$ is constant. This is preserved under differentiation with respect to t or h, which just shift the constant.

- The preceding fact allows us to derive relationships between critical exponents. For example,

$$C_{\text{sing}} \sim \frac{\partial^2 f}{\partial t^2} \sim |t|^{-\alpha} g_C(h/|t|^\Delta).$$

  Thus α is just the critical exponent α defined earlier.

- In general, we could let the exponents α and Δ depend on the sign of t, so that

$$C_\pm \sim |t|^{-\alpha_\pm} g_\pm(h/|t|^{\Delta_\pm}).$$

  However, there should be no singularity at $t = 0$ for finite h, where the two functions $C_\pm$ match. Hence $\alpha_+ = \alpha_-$ and $\Delta_+ = \Delta_-$, though g_+ and g_- may be distinct.

- Similarly, the magnetization has the form

$$m(t,h) \sim \frac{\partial f}{\partial h} \sim |t|^{2-\alpha-\Delta} g_m(h/|t|^{\Delta}).$$

  By similar reasoning to before, we may extract the leading behavior of $g_m(x)$ for high and low x. We thus find the relationships

$$\beta = 2 - \alpha - \Delta, \quad \delta = \Delta/\beta.$$

- Similarly, the susceptibility is

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} g_\chi(h/|t|^{\Delta})$$

  and similar arguments show that

$$\gamma = 2\Delta - 2 + \alpha.$$

We now step back and see what we've done. The homogeneity assumption has shown that:

- The singular parts of all critical quantities are homogeneous, with the same exponents above and below the transition.
- All of these quantities are described by the same gap exponent Δ. The two exponents α and Δ determine all critical exponents.
- The previous point implies that there are identities connecting the critical exponents. Using our above results, we find

$$\alpha + 2\beta + \gamma = 2, \quad \delta - 1 = \frac{\gamma}{\beta}$$

  which are called Rushbrooke's and Widom's identities.
- The exponent identities are completely consistent with experiment, as well as with our mean field approximation, even though the critical exponents differ widely between the two.

4.2 Correlation Lengths

So far, we haven't considered the divergence of the correlation length at the critical point. This can be related to our existing exponents by integrating the correlation function to yield a bulk susceptibility. However, it is more natural to 'start over' with a homogeneity assumption for the correlation length, from which we will recover our earlier results.

- We assume the correlation length ξ has the form

$$\xi(t,h) \sim |t|^{-\nu} g(h/|t|^{\Delta}).$$

  Furthermore, since the correlation length is the only length scale near the critical point, we assume that it is solely responsible for singular contributions to thermodynamic quantities.

- In particular, we may apply this assumption to the free energy $F \sim \log Z$. Since $\log Z$ is extensive, it is proportional to L^d. Since it is dimensionless, it must have the form

$$\log Z = \left(\frac{L}{\xi}\right)^d g_s + \left(\frac{L}{a}\right)^d g_a.$$

  The first term contributes to the singular behavior of the free energy; it is due to the large-scale fluctuations we investigated previously and is independent of a by assumption. The second term is regular at the critical point; for example, it could be the essentially constant contribution from phonons in our magnet example.

- Another way of thinking about the first term is that we can divide the sample into $(L/\xi)^d$ cells, within each of which we have independent large-scale fluctuations.

- Therefore, the singular part of the free energy density is

$$f_{\text{sing}}(t, h) \sim \frac{\log Z}{L^d} \sim \xi^{-d} \sim |t|^{d\nu} g_f(h/|t|^{\Delta}).$$

  This recovers our earlier homogeneity assumption, with Josephson's identity

$$2 - \alpha = d\nu.$$

  This is called a hyperscaling relation, as it depends on the spatial dimension d.

- Experimentally, the hyperscaling relation works for $d \leq 3$. For $d > 4$, we know that the results from mean field theory are correct; however, plugging in $\alpha = 0$ and $\nu = 1/2$, we find $d = 4$. Then the homogeneity assumption fails if mean field theory works.

- This is reasonable because the homogeneity assumption assumes that fluctuations are strong (i.e. that they give the dominant singular contribution to the free energy). But in $d > 4$, the dominant contribution instead comes from the mean field value, which gives $t^2 \propto \xi^{-4} > \xi^{-d}$.

We now consider the critical exponent η associated with the decay of the correlation function.

- As defined earlier, the magnetization correlation function falls off at the critical point as

$$G_m^c(\mathbf{x}) \sim \frac{1}{x^{d-2+\eta}}$$

  where $\eta = 0$ according to mean field theory, for $x \ll \xi$. Similarly, we can define an exponent for the decay of energy-energy correlations,

$$G_E^c(\mathbf{x}) = \langle \mathcal{H}(\mathbf{x})\mathcal{H}(0)\rangle \sim \frac{1}{x^{d-2+\eta'}}.$$

- By integrating over space, we may relate these correlation functions to bulk response functions near criticality; here, they are magnetic susceptibility and heat capacity, respectively. Then

$$\chi \sim \int_a^{\xi} d\mathbf{x}\, G_m^c(x).$$

  As discussed above, the integrand decays rapidly for $x > \xi$ and is cut off for $x < a$, where the field description breaks down. Here the upper bound determines the integral's value, giving

$$\gamma = (2 - \eta)\nu, \quad \alpha = (2 - \eta')\nu.$$

  The former result is called Fisher's identity.

- We could have guessed that the critical correlation functions were power laws ahead of time, because these are the only scale-invariant functions. At the critical point, we have perfect scale invariance; near it, we have scale invariance up to the length scale ξ. This behavior is characteristic of 'fractal geometry'.
- Indeed, if we added the assumption of dilation symmetry when constructing the Landau–Ginzburg probability, we could automatically be describing the critical point. However, it is difficult in general to see how dilation symmetry constrains the effective Hamiltonian. The exception is the case $d = 2$, where dilation symmetry implies conformal invariance.

The success of the scaling theory above supports the assumption that, close to the critical point, the correlation length ξ is the only important length scale; all microscopic lengths a drop out. The critical behavior is dominated by fluctuations that are self-similar up to the scale ξ.

4.3 Renormalization Group (Conceptual)

The idea of the renormalization group is to further coarse-grain our field. This generates a flow through 'theory space' which will allow us to understand continuous phase transitions. We first introduce the RG procedure conceptually. There are three components of each step.

- Coarse grain. Let a be the microscopic length scale that cuts off fluctuations of $\mathbf{m}(\mathbf{x})$. We decrease the resolution to ba with $b > 1$ by coarse-graining

$$\overline{m}_i(\mathbf{x}) = \frac{1}{(ab)^d} \int_{\text{near } \mathbf{x}} d\mathbf{x}'\, m_i(\mathbf{x}')$$

  where we integrate over a cell of volume $(ab)^d$.

- Rescale. The new coarse-grained field has fluctuations cut off at ba instead of a. To restore the original resolution, decrease all lengths scales, setting

$$\mathbf{x}_{\text{new}} = \frac{\mathbf{x}_{\text{old}}}{b}.$$

- Renormalize. The variations of fluctuations in the rescaled magnetization profile will generally be different from the original. This can be offset by scaling the field by a factor ζ.
- Combining the three steps above, the new magnetization field is

$$\mathbf{m}_{\text{new}}(\mathbf{x}_{\text{new}}) = \frac{1}{\zeta(ab)^d} \int_{\text{near } b\mathbf{x}_{\text{new}}} d\mathbf{x}'\, \mathbf{m}(\mathbf{x}').$$

  The RG transformation thus maps the variables $\mathbf{m}_{\text{old}}$ to $\mathbf{m}_{\text{new}}$. Since we know the probability distribution of $\mathbf{m}_{\text{old}}$, we can construct the distribution of $\mathbf{m}_{\text{new}}$ and hence the coarse-grained Hamiltonian $H_{\text{new}}[\mathbf{m}_{\text{new}}(\mathbf{x})]$. The old and new partition functions are numerically equal.

- If we think of the set of all possible coarse-grained Hamiltonians as a point in an infinite-dimensional 'theory space', then the RG transformation generates a flow through this theory space. As we flow, all of the couplings change and the system size and correlation length decrease. However, everything about the long-distance physics remains exactly the same.

We now heuristically apply this procedure to the Landau–Ginzburg Hamiltonian.

- Since the critical point occurs when the parameters t and h are tuned to zero, we will assume that we only have to look at the RG flow of t and h near this point.
- Since the transformation only involves changes at the shortest length scales, it cannot produce singularities, so the renormalized parameters are analytic functions of the original ones.
- Near the critical point, we may expand in a Taylor series,

  $$t_b(t,h) = A(b)t + B(b)h + \dots, \quad h_b(t,h) = C(b)t + D(b)h + \dots.$$

  We know there are no constant terms, because the critical point is a fixed point.
- Rotational symmetry means that the probability is invariant under $m \to -m$, $h \to -h$, $t \to t$. This eliminates cross-terms between t and h, so

  $$t_b(t,h) = A(b)t + \dots, \quad h_b(t,h) = D(b)h + \dots.$$

- Since $A(1) = 1$ and $A(b_1)A(b_2) \approx A(b_1 b_2)$, $A(b)$ and similarly $D(b)$ must be powerw,

  $$t' = t_b = b^{y_1}t + \dots, \quad h' = h_b = b^{y_h}h + \dots.$$

  This holds for all points near the critical point. Since RG flow reduces the correlation length, it must take us away from criticality, so y_t and y_h must be positive.
- By construction, we have $Z = Z'$, so the free energy densities are related by

  $$V f(t,h) = V' f(t',h') \quad \to \quad f(t,h) = b^{-d} f(b^{y_t}t, b^{y_h}h).$$

  To recover the homogeneity assumption, we choose $b = t^{-1/y_t}$, giving

  $$f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t}).$$

  Therefore $2 - \alpha = d/y_t$ and $\Delta = y_h/y_t$. We also see that there may be separate functions for positive and negative t, as the RG flow can't change the sign of t.
- We also know that correlation lengths are reduced by a factor of b, so

  $$\xi(t,h) = b\xi(b^{y_t}t, b^{y_h}h) = t^{-1/y_t}\xi(1, h/t^{y_h/y_t}) \sim t^{-\nu}$$

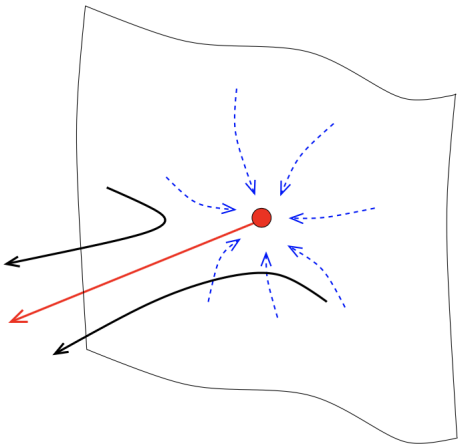
  at $h = 0$. Then we conclude $\nu = 1/y_t$. Combining with the previous result recovers the hyperscaling identity. (This is strange, as we know hyperscaling fails for $d > 4$. We will explain why at the end of this section.)
- By differentiating the free energy, we can obtain other bulk quantities such as magnetization. Alternatively, we can compute it directly using what we know about the RG flow,

  $$m(t,h) = -\frac{1}{V}\frac{\partial \log Z(t,h)}{\partial h} = -\frac{1}{b^d V'}\frac{\partial \log Z'(t',h')}{b^{-y_h}\partial h'} = b^{y_h - d} m(b^{y_t}t, b^{y_h}h).$$

  Then we find $\beta = (y_h - d)/y_t$, in agreement with our previous result.
- Generally, we see that everything scales with gap exponent $\Delta = y_h/y_t$. Moreover, let the extra exponent in front be b^{y_X} for a quantity X. Then if X and F are conjugate variables (i.e. they appear in the Hamiltonian as $\int d\mathbf{x}\, FX$) then $y_X = y_F - d$, where $F' = b^{y_F}F$ under RG. This subsumes our above calculation as a special case.

We have thus recovered all of our scaling results from a conceptual RG analysis.

4.4 Renormalization Group (Formal)



We now analyze the RG procedure more formally, justifying our earlier assumptions.

- We begin with the most general Hamiltonian allowed by symmetries. For example, given rotational symmetry,

$$\beta H = \int d\mathbf{x} \left(\frac{t}{2} m^2 + u m^4 + v m^6 + \cdots + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \cdots \right).$$

  A system is thus specified by a point in an infinite-dimensional parameter space.

- Using the three steps described above, we construct the change of variables $\mathbf{m}(\mathbf{x}) \to \mathbf{m}'(\mathbf{x}')$, then construct the new probability distribution $P'[\mathbf{m}'(\mathbf{x}')]$ and hence a new Hamiltonian $H'[\mathbf{m}'(\mathbf{x}')]$.

- Since rotational symmetry is preserved, the rescaled Hamiltonian must also be described by a point in the parameter space S, so we have a map $S' = R_b(S)$.

- Since RG flow scales the correlation length, fixed points of the RG flow must have either zero correlation length (corresponding to independent fluctuations, which occur for zero or infinite temperature) or infinite correlation length (corresponding to critical points).

- We may linearize about a fixed point S^* so that, under an RG transformation, the displacement from the fixed point becomes

$$\delta S'_\alpha = (R_b^L)_{\alpha\beta} \delta S_\beta, \quad (R_b^L)_{\alpha\beta} = \left. \frac{\partial S'_\alpha}{\partial S_\beta} \right|_{S^*}.$$

- Generically, each of the R_b will have a complete set of eigenvectors. We know that

$$R_b^L R_{b'}^L = R_{bb'}^L = R_{b'}^L R_b^L$$

  so that the R_b commute with each other. Then we may simultaneously diagonalize them; by the same argument as before, we conclude that the eigenvalues are

$$\lambda(b)_i = b^{y_i}$$

  where the eigenvectors $\mathcal{O}_i$ correspond to operators. We call the y_i the anomalous dimensions. This differs from the usual usage; in the standard convention the anomalous dimensions are the differences of the dimensions vs. those at the Gaussian fixed point.

- Therefore, near the fixed point, we have

$$S^* + \sum_i g_i \mathcal{O}_i \quad \to \quad S^* + \sum_i g_i b^{y_i} \mathcal{O}_i.$$

  If $y_i > 0$, then g_i increases under scaling, and we say $\mathcal{O}_i$ is a relevant operator. If $y_i < 0$, then g_i decreases and $\mathcal{O}_i$ is irrelevant; it vanishes under RG flow and does not affect the long-distance physics. Finally, if $y_i = 0$, then $\mathcal{O}_i$ is marginal, and higher-order terms are needed to track its behavior.

- The subspace spanned by the irrelevant operators is called the basin of attraction of the fixed point S^*. The correlation length is infinite for all points in the basin, so they are all 'critical points' for some system.

- Mathematically, RG flow is as general as phase space flow for an arbitrary dynamical system. The basins of attraction are generally curved surfaces, not planes, and there may be many fixed points. Exotic physical systems such as spin glasses can even have limit cycles. This physically corresponds to a *discrete* scale invariance, and a pure imaginary critical exponent.

Now consider the Landau–Ginzburg Hamiltonian with no external field.

- When we adjust T, we go through a critical point when the coefficient of m^2 goes through zero. This is a line through the abstract parameter space S, which implies that there is a basin of attraction of codimension one, and hence only one relevant operator, m^2.

- If we perform RG flow before performing this analysis, then the line will be shifted; it will still intersect the basin, but at a point closer to its fixed point.

- Different materials have different values for higher-order terms in the Landau–Ginzburg Hamiltonian, and generally will also go along different curves through S. But as long as we coarse-grain beforehand, all of them will go through the fixed point! This explains universality.

- All points with $t \neq 0$ eventually flow to a low-temperature or high-temperature fixed point, characterized by zero correlation length (independent spins).

- If an experiment shows that a critical point can be reached by tuning N parameters, we need to find an RG fixed point with N relevant operators.

  For example, the Landau–Ginzburg critical point is reached by tuning one parameter, $T \to T_c$, because m^2 is relevant. On the other hand, if we include the possibility of external fields, we must also count tuning the external field to zero.

- RG flow from a point near the fixed point will take irrelevant operators to zero. The leading singularity in the correlation length is given by the relevant operator with the greatest anomalous dimension; the other relevant operators each contribute a gap exponent.

- Universality is a blessing in the IR and a curse in the UV: it tells us that it is difficult to learn anything about UV physics from IR physics.

4.5 The Gaussian Model

As an explicit example of the renormalization group, we consider the Gaussian model, i.e. the free field theory obtained by keeping quadratic terms in the Landau–Ginzburg Hamiltonian. We first solve it directly, then apply RG ideas.

- The partition function is

$$Z = \int \mathcal{D}\mathbf{m}(\mathbf{x}) \exp\left(-\int d\mathbf{x}\, \frac{t}{2} m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots - \mathbf{h}\cdot\mathbf{m}\right).$$

  This model is only well-defined for $t > 0$, so we only have one side of a phase transition.

- We can directly calculate Z by Fourier transform. We place the system in a box of volume V with periodic boundary conditions, so the momentum values $\mathbf{q}$ are discretized in units of $2\pi/L$ and confined to a Brillouin zone whose shape is determined by the underlying lattice.

- We take the Fourier transform convention

$$\mathbf{m}(\mathbf{q}) = \int d\mathbf{x}\, e^{i\mathbf{q}\mathbf{x}} \mathbf{m}(\mathbf{x}), \quad \mathbf{m}(\mathbf{x}) = \sum_{\mathbf{q}} \frac{e^{-i\mathbf{q}\mathbf{x}}}{V} \mathbf{m}(\mathbf{q}) = \int đ\mathbf{q}\, e^{-i\mathbf{q}\mathbf{x}} \mathbf{m}(\mathbf{q}).$$

  We have to be careful with volume factors, as the volume changes under RG flow. In the last equality, we're using the standard transformation between a sum and an integral in the continuum limit; the integration range is the Brillouin zone.

- Under this convention,

$$\int d\mathbf{x}\, \mathbf{m}(\mathbf{x})^2 = \sum_{\mathbf{q}} \frac{\mathbf{m}(\mathbf{q})\cdot\mathbf{m}(-\mathbf{q})}{V} = \sum_{\mathbf{q}} \frac{|\mathbf{m}(\mathbf{q})|^2}{V}.$$

  Higher order terms just pick up momentum factors, giving

$$\beta H = \sum_{\mathbf{q}} \left(\frac{t + Kq^2 + Lq^4 + \cdots}{2V}\right) |\mathbf{m}(\mathbf{q})|^2 - \mathbf{h}\cdot\mathbf{m}(\mathbf{q}=0).$$

- Now we evaluate Z. The path integral measure $\mathcal{D}(\mathbf{m}(\mathbf{x}))$ is proportional to the product of nN individual differentials $dm_i(\mathbf{x}_j)$, where N is the number of lattice sites and n is the number of components of $\mathbf{m}$. Each of these modes picks up a Jacobian factor of $1/\sqrt{V}$, so

$$Z = \int \left(\prod_{\mathbf{q}} V^{-n/2}\, d\mathbf{m}(\mathbf{q})\right) e^{-\beta H}.$$

- Each momentum integral is a Gaussian, and for $\mathbf{q} \neq 0$, the Jacobian factors cancel out. The $\mathbf{q} = 0$ integral is special because it picks up the $\mathbf{h}\cdot\mathbf{m}$ shift, giving an extra factor of $\exp(Vh^2/2t)$. The final result is

$$Z = \exp\frac{Vh^2}{2t} \prod_{\mathbf{q}} \left(\frac{2\pi}{t + Kq^2 + Lq^4 + \cdots}\right)^{n/2}.$$

We now see the scaling behavior that our result yields.

- The free energy density is then

$$f = -\frac{\log Z}{V} = \frac{n}{2}\int_{\mathrm{BZ}} đ\mathbf{q}\,\log(t + Kq^2 + Lq^4 + \cdots) - \frac{h^2}{2t}.$$

  We see that we've correctly kept track of volume factors, since f is independent of V, making the free energy extensive.

- We now must integrate over the Brillouin zone. The integrand has no singularities for large q, so this region does not contribute to the singular behavior of f. Hence we may deform the Brillouin zone to a hypersphere of radius $\Lambda \sim \pi/a$, so

$$f_{\mathrm{sing}}(t,h) = \frac{n}{2}K_d\int_0^\Lambda dq\, q^{n-1}\log(t + Kq^2 + Lq^4 + \cdots) - \frac{h^2}{2t}$$

  where $K_d = \Omega_d/(2\pi)^d$.

- To extract the leading singular behavior of the integral (in t), note that under rescaling q by a factor of $\sqrt{t/K}$, we get

$$\left(\frac{t}{K}\right)^{d/2}\int_0^{\Lambda\sqrt{K}/\sqrt{t}} dx\, x^{d-1}\log(t + tx^2 + Lt^2x^4/K^2 + \cdots).$$

  Since the UV isn't the important part, we can ignore the fact that the bounds on the integral have changed, so all its t-dependence is in the integrand.

- The integrand expands as

$$\log t + \log(1 + x^2 + Ltx^4/K^2 + \ldots) = \log t + O(t)$$

  so that its leading singular behavior in t is $\log t$.

- Since $\log t$ is like a zeroth power, it doesn't contribute to critical exponents. Ignoring it,

$$f_{\mathrm{sing}}(t,h) \sim \left(\frac{t}{K}\right)^{d/2} - \frac{h^2}{2t} = t^{2-\alpha}g_f(h/t^\Delta)$$

  where

$$\alpha_+ = 2 - d/2, \quad \Delta = 1/2 + d/4.$$

  Note that this expression only holds for $t > 0$, as the model is unstable for $t < 0$. We have thus recovered the scaling hypothesis, as expected.

- The resulting critical exponents match those of mean field theory when $d = 4$, but surprisingly are incorrect for $d > 4$. We explain this anomaly below.

Next, we repeat the analysis using the renormalization group.

- To implement the coarse graining, we integrate out the modes with momentum between Λ/b and Λ. Since these modes don't mix with the modes we keep, this step is trivial, and simply adds a nonsingular contribution $\delta f_b(t)$ to the free energy, which we ignore.

- Next, we change variables to the primed variables,

$$\mathbf{x}' = \frac{\mathbf{x}}{b}, \quad \mathbf{q}' = b\mathbf{q}, \quad \phi(\mathbf{x}) = \zeta\phi'(\mathbf{x}'), \quad \phi(\mathbf{k}) = z\phi'(\mathbf{k}')$$

  where the real space and momentum space renormalization factors ζ and z are different. Since the RG step was trivial, this is the only step that changes the couplings.

- In momentum space, we have

$$t' = z^2 b^{-d} t, \quad h' = zh, \quad K' = z^2 b^{-d-2} K, \quad L' = z^2 b^{-d-4} L, \ldots.$$

  We see $t = h = 0$ is a fixed point, as expected. To make the fluctuations scale-invariant, we choose $K' = K$ and hence $z = b^{1+d/2}$, which makes higher-order gradients $L, \ldots$ irrelevant.

- We conclude that t and h scale as

$$t' = b^2 t, \quad h' = b^{1+d/2} h$$

  giving critical exponents of $y_t = 2$ and $y_h = 1 + d/2$, respectively. Applying our earlier formulas recovers the scaling exponents α and Δ derived earlier. We see that there is an RG fixed point, called the Gaussian fixed point, where K is the only nonzero coefficient. Near this fixed point the critical exponents are those of mean field theory.

- Next, we consider higher-order terms. In general, we will treat their effect perturbatively; working to zeroth order, the higher-order terms have no effect on the first step of RG. In real space, the Gaussian fixed point is

$$\beta H^* = \frac{K}{2} \int d^d\mathbf{x}\, (\nabla m)^2, \quad K \to K' = b^{d-2}\zeta^2 K$$

  which sets $\zeta = b^{1-d/2}$. In general the real space and momentum space factors differ by a factor of b^d, because $\phi(\mathbf{k} = 0) \sim \int d^d\mathbf{x}\, \phi(\mathbf{x})$. When we talk about scaling dimensions without further qualification, we usually mean real space.

- Now, an m^n term will pick up a factor of $b^d \zeta^n$, giving a scaling dimension

$$y_n = n - d(n/2 - 1).$$

  The cases $n = 1$ and $n = 2$ recover y_h and y_t, as expected. The case $n = 3$ is ruled out by spherical symmetry. However, $n = 4$ gives $y_u = 4 - d$, and hence is relevant for $d < 4$.

Thus for $d < 4$, the Gaussian fixed point does not describe the critical behavior, as it has more than one relevant operator. In the next section, we will perform a more careful analysis which shows that another fixed point, the Wilson–Fisher fixed point, characterizes the critical behavior.

4.6 ∗ Appendix

In this section we collect some extra notes on the material above.

Note. Where does the temperature appear in RG flow? We have set $\beta = 1$, or equivalently absorbed it into the coupling constants. Then RG flow can't tell us anything about temperature; it only gives us a flow in the coefficients of βH. Moreover, the temperature can't be recovered from βH alone; it is an additional layer of structure independent of our existing formalism.

Despite this, we sometimes loosely speak of RG flow changing the temperature. This is because, for the Landau–Ginzburg Hamiltonian, the coefficient $t(T)$ of ϕ^2 functions as a 'thermometer', with known dependence on T near the critical point. Then when RG flow changes t, we can think of it as effectively changing the temperature to T' so that $t'(T') = t(T)$. That is, our coarse-grained system looks close to what our original system would have looked like at temperature T'. It's not identical, but for the purposes of calculating critical exponents such as ν, we just need our heuristic temperature to be related to the real one in a nonsingular way.

Note. How is the renormalization factor ζ chosen? RG flow connects points in theory space with equivalent long-range dynamics. Trivially rescaling the field also defines a flow connecting equivalent theories. The full RG flow consists of a combination of these two flows, and can look drastically different depending on the choice of ζ.

In practice, the factor ζ is determined by demanding that the experimentally observed critical point is an RG fixed point. While any other choice of ζ would also be valid, it would not necessarily be useful. In the case of Landau–Ginzburg, we choose ζ to fix the coefficient of $(\nabla m)^2$ to describe the Gaussian fluctuations near the critical point.

Note. The Gaussian model gives the wrong results for $d > 4$ because we've neglected the um^4 term as irrelevant, but it has a strong, non-analytic effect on the expectation value of m. Consider the scaling for $t < 0$ and $h = 0$ with u included. Then the scaling relation for m is

$$m(t, u) = b^{1-d/2} m(b^2 t, u b^{4-d})$$

where $\zeta = b^{1-d/2}$ is the real-space renormalization factor ζ. Then choosing $b^2 t = 1$ gives

$$m(t, u) = t^{(d-2)/4} m(-1, u t^{(d-4)/2})$$

and ignoring the u term yields $\beta = (d-2)/4$, which is only correct in $d = 4$. If $m(-1, u)$ were analytic in u, this would be valid because Taylor expanding would yield subleading powers of t. However, we instead have $m \propto 1/\sqrt{u}$ by mean field theory, so the leading singularity is

$$m(t, u) \sim t^{(d-2)/4} \left(u t^{(d-4)/2} \right)^{-1/2} = u^{-1/2} t^{1/2}$$

which gives $\beta = 1/2$ as desired. By accounting for the 'dangerously irrelevant' behavior of u, we can recover all other mean field theory critical exponents. In the case $d = 4$, we don't need to do this because u is marginal.

- Our argument does not apply to $t > 0$ and $h = 0$, where $m = 0$. This is okay, because β is only defined for $t < 0$.
- Consider $t = 0$ and nonzero h. Then the magnetization diverges as $h^{1/\delta}$. We can run a similar scaling argument, fixing $h = 1$ and setting $t = 0$, to recover the mean field result $\delta = 3$.
- We can integrate the magnetization to get the free energy, whose scaling is also affected, modifying the critical exponent α.

- Consider the zero-field susceptibility for $t > 0$, which diverges as t^{γ_+}. The naive Gaussian model result is correct here because for a small h, $m \sim h +$ terms analytic in u. Continuing around the critical point using nonzero h gives $\gamma_+ = \gamma_-$.
- Generally, the operator um^4 only affects the free energy, not the correlation functions, so η and $\nu_\pm$ remain correct. However, the link between the free energy and correlation functions breaks down, so hyperscaling fails.

Note. The scaling arguments above can be reproduced by a sophisticated 'dimensional analysis'.

- Near a fixed point, consider a perturbation of the form $g\mathcal{O}$ so that $\mathcal{O}$ is an eigenvector of the RG flow, so under an RG step
$$g \to \zeta^{\Delta_g} g.$$
Here, g is a dimensionful coupling. Near the Gaussian fixed point Δ_g will coincide with the mass dimension of g, also called the engineering dimension. But in general, the two will differ.
- Motivated by engineering dimensional analysis, we define the scaling dimension of $\mathcal{O}$ as
$$\Delta_{\mathcal{O}} = d - \Delta_g.$$
Again, this coincides with the mass dimension of $\mathcal{O}$ at the Gaussian fixed point; the operator $\mathcal{O}$ is relevant if $\Delta_{\mathcal{O}} < d$.
- As shown in the notes on Quantum Field Theory, correlators of $\mathcal{O}$'s decay as
$$\langle \mathcal{O}(0)\mathcal{O}(\mathbf{x})\rangle \sim \frac{1}{r^{2\Delta_{\mathcal{O}}}}.$$
One might think that by regular dimensional analysis, this proves that $\Delta_{\mathcal{O}}$ must coincide with the engineering dimension. This is false because the cutoff scale a is implicitly present; the correlation function essentially depends on factors of a/r.
- Note that in general, the field $\phi(\mathbf{x})$ itself won't be an RG eigenvector. Moreover, RG eigenvalues don't add under multiplication like dimensions do, i.e. generally $\Delta_{\mathcal{O}_1\mathcal{O}_2} = \Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2}$ is false. However, both of these statements are approximately true at weak coupling, so we'll ignore these subtleties.
- Applying this to the field $\phi(\mathbf{x})$, we define
$$\Delta_\phi = \frac{d - 2 + \eta}{2}.$$
Since ϕ is not an RG eigenvalue, Δ_ϕ really means the lowest $\Delta_{\mathcal{O}}$ for any $\mathcal{O}$ with a ϕ component, as this operator dominates at long distances. At the Gaussian fixed point, $\eta = 0$.
- This logic helps us derive scaling relations. Since the free energy contains a $B\phi$ term,
$$\Delta_B = d - \Delta_\phi = d - 2 + \eta.$$
On the other hand, we have $\phi \sim B^{1/\delta}$, so
$$\delta = \frac{\Delta_B}{\Delta_\phi} = \frac{d + 2 - \eta}{d - 2 + \eta}.$$

- We also have $\phi \sim t^\beta$, where we know $\xi \sim t^{-\nu}$ and $\Delta_\xi = -1$ since it is a physical length. Then

$$\Delta_t = \frac{1}{\nu}, \quad \beta = \nu\Delta_\phi = \frac{(d-2+\eta)\nu}{2}.$$

- Finally, the susceptibility is

$$\chi = \frac{\partial\phi}{\partial B} \sim t^{-\gamma}, \quad \Delta_\phi - \Delta_B = -\frac{\gamma}{\nu}, \quad \gamma = \nu(2-\eta).$$

  We have introduced the intermediate quantities Δ_B and Δ_t, where the gap exponent Δ defined earlier is $\Delta = \Delta_B/2\Delta_t$.

- The fourth and final critical exponent relation is the hyperscaling relation. The free energy density f scales as

$$f \sim \xi^{-d} \sim t^{d\nu}, \quad c \sim t^{d\nu-2}, \quad \alpha = 2 - d\nu.$$

Note. The emergence of rotational symmetry. Consider a square lattice in $d = 2$ dimensions. We have a dihedral symmetry group D_4. The most relevant term that breaks $SO(2)$ symmetry but not D_4 symmetry is $\phi(\partial_1^4 + \partial_2^4)\phi$, but its scaling dimension near the Gaussian fixed point is $\Delta_g = -2$. Hence the term is irrelevant; under coarse-graining, full rotational symmetry emerges.

Note. So far we've been assuming $\mathbb{Z}_2$ symmetry, $\phi \to -\phi$. As we've seen, for a magnet we can break this symmetry by turning on an external field, which provides a relevant operator; then to reach the critical point we also need to tune the external field to zero. For the liquid/gas critical point, there is no manifest $\mathbb{Z}_2$ symmetry. Correspondingly, we must tune two parameters, i.e. the pressure and the temperature. Once we break $\mathbb{Z}_2$ symmetry we might worry about a ϕ^3 term, which would also be relevant, but it can be removed by a field redefinition.

5 Perturbative Renormalization Group

5.1 Expectation Values

We begin by showing how perturbation theory works for statistical fields.

- Consider the Gaussian model,

$$\beta H_0 = \int đ\mathbf{q}\, \frac{t + Kq^2 + Lq^4 + \cdots}{2} |\mathbf{m}(\mathbf{q})|^2.$$

  One possible perturbation is the m^4 interaction

$$U = u \int d\mathbf{x}\, m(\mathbf{x})^4 = u \int đ\mathbf{q}_1\, đ\mathbf{q}_2\, đ\mathbf{q}_3\, m_\alpha(\mathbf{q}_1) m_\alpha(\mathbf{q}_2) m_\beta(\mathbf{q}_3) m_\beta(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3).$$

- The unperturbed Hamiltonian has correlation function

$$\langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}') \rangle_0 = \frac{\delta_{\alpha\beta} đ\!\!\!/(\mathbf{q} + \mathbf{q}')}{t + Kq^2 + Lq^4 + \cdots}.$$

- More generally, an n-point correlation function can be expanded as

$$\left\langle \prod_i m_i \right\rangle_0 = \text{sum over pairwise contractions}$$

  by Wick's theorem. For example, the four-point correlation is the sum of three terms, each a product of two two-point correlators.

- In the presence of an interaction U, we can calculate the expectation value of any operator O in perturbation theory as

$$\begin{aligned}
\langle O \rangle = \frac{\int \mathcal{D}\mathbf{m}\, O e^{-\beta H_0 - U}}{\int \mathcal{D}\mathbf{m}\, e^{-\beta H_0 - U}} &= \frac{\int \mathcal{D}\mathbf{m}\, e^{-\beta H_0} O(1 - U + U^2/2 - \cdots)}{\int \mathcal{D}\mathbf{m}\, e^{-\beta H_0}(1 - U + U^2/2 - \cdots)} \\
&= \frac{\langle O \rangle_0 - \langle OU \rangle_0 + \langle OU^2 \rangle_0/2 - \cdots}{1 - \langle U \rangle_0 + \langle U^2 \rangle_0/2 - \cdots} \\
&= \left(\langle O \rangle_0 - \langle OU \rangle_0 + \langle OU^2 \rangle_0/2 - \cdots\right) \left(1 + \langle U \rangle_0 + \langle U \rangle_0^2 - \frac{1}{2}\langle U^2 \rangle_0 + \cdots\right)
\end{aligned}$$

  where $\langle \cdot \rangle_0$ denotes an expectation value under the unperturbed Hamiltonian, and we inverted the denominator by power series.

- We group the terms by powers of U, so that

$$\langle O \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle OU^n \rangle_0^c$$

  where $\langle \cdot \rangle^c$ is the connected average. We have already seen the case $n = 1$, the 'connected correlation function'. Such an expansion in terms of connected diagrams also appears for correlation functions in quantum field theory.

Example. Let O be the two-point correlation function $\langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}')\rangle$, under the um^4 perturbation. The first order shift is

$$-\langle OU\rangle_0^c = \langle O\rangle_0\langle U\rangle_0 - \langle OU\rangle_0.$$

Explicitly, we have

$$\langle OU\rangle_0 = u\int đ\mathbf{q}_1\, đ\mathbf{q}_2\, đ\mathbf{q}_3\, \langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}') m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0$$

and

$$\langle O\rangle_0\langle U\rangle_0 = u\int đ\mathbf{q}_1\, đ\mathbf{q}_2\, đ\mathbf{q}_3\, \langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}')\rangle_0\, \langle m_i(\mathbf{q}_1) m_i(\mathbf{q}_2) m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0.$$

We may evaluate both of these expectation values using Wick's theorem, summing over pairwise contractions. The contractions in the former term that don't link together the O and U fields are canceled by the ones in the second. Then the only terms that count are the ones that connect the O and U fields, justifying the 'connected correlation function' name. More generally, for $\langle OU^n\rangle_0^c$, the only terms that count are the ones that connect all $n+1$ operators together.

Now, in our current situation, we find two types of terms. We may contract m_α and m_β to m's with the same index (4 ways), giving a contribution of

$$4u\langle m_\alpha(\mathbf{q}) m_i(\mathbf{q}_1)\rangle_0\langle m_\beta(\mathbf{q}') m_i(\mathbf{q}_2)\rangle_0\langle m_j(\mathbf{q}_3) m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0$$

The index structure is $\delta_{\alpha i}\delta_{\beta i}\delta_{jj} = n\delta_{\alpha\beta}$. The momentum delta functions perform the $\mathbf{q}_1$ and $\mathbf{q}_2$ integrals, giving

$$4u\frac{n\delta_{ab}đ(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{đ\mathbf{q}_3}{t+Kq_3^2}.$$

Above, we have set all unperturbed coefficients except for t and K to zero, for brevity. We also have 8 terms where m_α and m_β are contracted to different indices. The structure is identical, but with an overall factor of 8 instead of $4n$. Therefore, the propagator is

$$\langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}đ(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left[1-\frac{4u(n+2)}{t+Kq^2}\int\frac{đ\mathbf{k}}{t+Kk^2}\right]$$

to first order in u.

Note. The first-order propagator above has many of the features of the unperturbed propagator. More generally, we claim that the exact propagator must have the form

$$\langle m_\alpha(\mathbf{q}) m_\beta(\mathbf{q}')\rangle = \delta_{\alpha\beta}đ(\mathbf{q}+\mathbf{q}')S(q).$$

The delta function is imposed by momentum conservation, i.e. translational symmetry. The $S(q)$ factor only depends on the norm of q by spatial rotational symmetry. The $\delta_{\alpha\beta}$ is imposed by rotational symmetry in the internal space, as this is the only rotationally invariant tensor.

Note. In the high temperature phase, we have $S(q=0)=\chi$ by definition. Expanding to first order in u, we find

$$\chi^{-t}(t) = t + uI(t) + O(u^2),\quad I(t) = 4(n+2)\int\frac{đ\mathbf{k}}{t+Kk^2}.$$

Therefore, for positive u, the critical point occurs at a lower temperature, as the um^4 term provides 'additional stability'. In particular, the critical temperature satisfies

$$t_c = -uI(t_c) + O(u^2) = -uI(t=0) + O(u^2)$$

where we replaced $I(t_c)$ with $I(0)$ since we're working to first order in u. Now,

$$I(t=0) \propto \int \frac{đ\mathbf{k}}{k^2} \propto \frac{\Lambda^{d-2}}{d-2}$$

so we have a finite, $O(u)$ shift to the critical temperature for $d > 2$. In $d < 2$, the integral is IR divergent, as expected from the Mermin–Wagner theorem.

We may also extract the critical behavior. Using our previous expression, we find

$$\chi^{-1}(t) - \chi^{-1}(t_c) \sim (t - t_c)\left[1 - uA\int \frac{đ\mathbf{k}}{k^2(k^2 + (t-t_c)/K)}\right]$$

where we have absorbed several constants into A. For high momenta, the k^4 term in the denominator dominates, giving a Λ^{d-4} term. However, the problem also contains the momentum scale $\sqrt{(t-t_c)/K}$, so generically

$$\int \frac{đ\mathbf{k}}{k^2(k^2 + (t-t_c)/K)} \sim \Lambda^{d-4} + \left(\frac{K}{t-t_c}\right)^{(4-d)/2}.$$

For $d < 4$, the second term dominates, so the perturbation series breaks down; we see that the critical behavior is modified by the quartic term. Physically, the reason is that the dimensionless parameters we use to construct the perturbation series are ua^{4-d}/K^2 and $u\xi^{4-d}/K^2$. In the case $d < 4$, the latter diverges, so the perturbation series breaks down. This is just another manifestation of the problems we run into below the upper critical dimension.

5.2 Perturbative RG

We can use perturbation theory to calculate the RG flow of the Landau–Ginzburg Hamiltonian. We write the quartic term as $U = u\int d\mathbf{x}\, m^4$ and expand in u.

- To coarse-grain, we split the field into components

  $$\mathbf{m}(\mathbf{q}) = \mathbf{m}'(\mathbf{q}) + \boldsymbol{\sigma}(\mathbf{q})$$

  where $\mathbf{m}'$ has support for $q < \Lambda/b$, and $\boldsymbol{\sigma}$ has support for $\Lambda/b < q < \Lambda$.

- The partition function is then

  $$Z = \int \mathcal{D}\mathbf{m}'(\mathbf{q})\mathcal{D}\boldsymbol{\sigma}(\mathbf{q})\, \exp\left[-\int_0^\Lambda đ\mathbf{q}\, \left(\frac{t+Kq^2}{2}\right)(|m'(\mathbf{q})|^2 + |\sigma(\mathbf{q})|^2) - U[\mathbf{m}'(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q})]\right].$$

  To perform the coarse-graining, we integrate out $\boldsymbol{\sigma}$. We see that the Gaussian terms for $\mathbf{m}'$ and $\boldsymbol{\sigma}$ don't interact, but the m^4 term mixes them up, giving nontrivial behavior.

- More precisely, the integral we wish to do is an unnormalized expectation value,

  $$I[\mathbf{m}'] = \int \mathcal{D}\boldsymbol{\sigma}(\mathbf{q})\, \exp\left[-\int_{\Lambda/b}^\Lambda đ\mathbf{q}\, \left(\frac{t+Kq^2}{2}\right)|\sigma(\mathbf{q})|^2 - U[\mathbf{m}'(\mathbf{q}), \boldsymbol{\sigma}(\mathbf{q})]\right] = Z_0\langle e^{-U[\mathbf{m}']}\rangle_0$$

  where $Z_0 = \langle 1\rangle_0$ and $\langle\cdot\rangle_0$ is averaging with respect to the Gaussian Hamiltonian for $\boldsymbol{\sigma}$.

- Taking the log, Z_0 does not depend on $\mathbf{m}'$ and has finite bounds of integration. Then it gives a finite additive contribution to the free energy, which we ignore. Then the coarse-grained Hamiltonian is

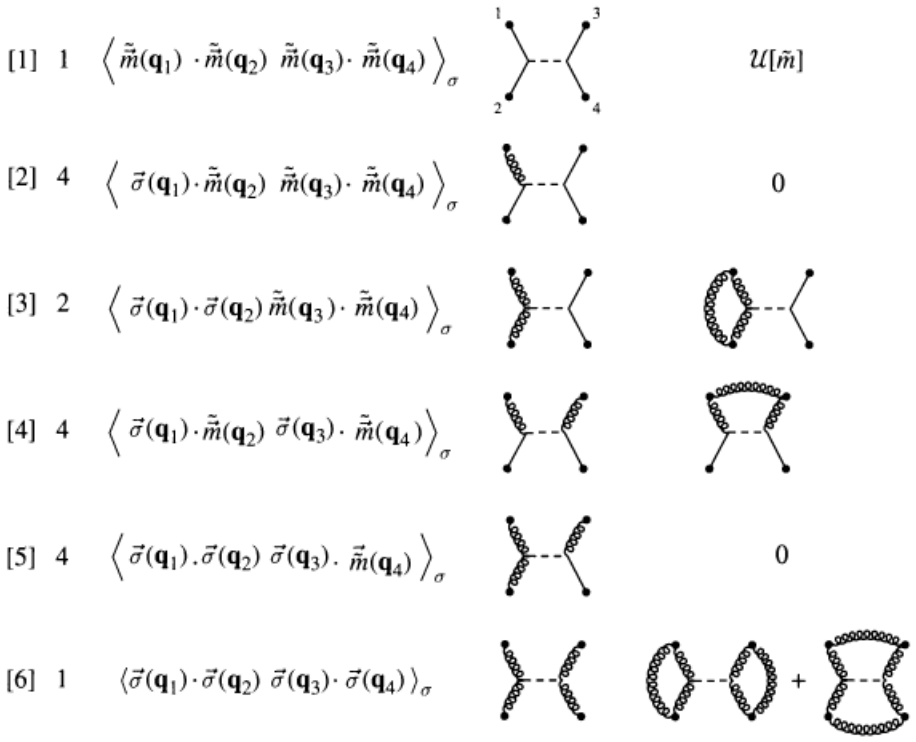
$$\boxed{\beta H'[\mathbf{m}'] = \int_0^{\Lambda/b} đ\mathbf{q} \left(\frac{t + Kq^2}{2}\right) |m'(\mathbf{q})|^2 - \log\langle e^{-U[\mathbf{m}']}\rangle_0.}$$

  where we have

$$-\log\langle e^{-U}\rangle_0 = \langle U\rangle_0 - \frac{1}{2}\left(\langle U^2\rangle_0 - \langle U\rangle_0^2\right) + \cdots + \frac{(-1)^{\ell+1}}{\ell!}(\ell^{\text{th}} \text{ cumulant of } U) + \cdots .$$

  Diagrammatically, the cumulants count only connected diagrams, as seen before.

We have thus set up the perturbative RG flow of the Landau–Ginzburg Hamiltonian. We first compute the RG flow to first order. We draw uncontracted diagrams in the first column, then the results of contractions in the second.



- At first order, the effective Hamiltonian picks up a term $-\langle U\rangle_0$. Explicitly,

$$\langle U\rangle_0 = u\int đ\mathbf{q}_1\, đ\mathbf{q}_2\, đ\mathbf{q}_3\, đ\mathbf{q}_4\, \not{\delta}(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\,\langle \mathbf{m}(\mathbf{q}_1)\cdot\mathbf{m}(\mathbf{q}_2)\,\mathbf{m}(\mathbf{q}_3)\cdot\mathbf{m}(\mathbf{q}_4)\rangle_0 .$$

  Each of the $\mathbf{m}$ factors is expanded as $\mathbf{m} = \mathbf{m}' + \boldsymbol{\sigma}$, giving 16 terms.

- Diagrammatically, we have a four-point vertex; applying Wick's theorem contracts pairs of legs. We now have two fields: the 'external source' $\mathbf{m}'$, drawn as a straight line, and $\boldsymbol{\sigma}$, drawn as a curly line. The vertex also has a nontrivial vector structure (it is the product of two scalars which are obtained by contracting vectors), which we draw with a dotted line.

- There are six types of vertices, as shown above. Type [1] gives back $U[\mathbf{m}']$, the expected quartic term. Types [2] and [5] are odd in $\boldsymbol{\sigma}$ and hence average to zero; they have no nonzero contractions. Type [6] has no dependence on $\mathbf{m}'$ and hence just contributes an additive constant to the free energy, which we ignore.

- We get nontrivial contributions from type [3] and type [4]. The resulting integrals are similar to those of the previous section; as before, a closed σ loop gives a factor of n. Since both are proportional to m^2, the result is a renormalization of t,

$$\widetilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{đ\mathbf{k}}{t + Kk^2} \equiv t + uI(t).$$

  However, K and u are unaffected.

- Explicitly, we are using the following Feynman rules for the n^{th} order term.
  - Draw all possible diagrams with n vertices. For each vertex, multiply by u.
  - Contract pairs of $\boldsymbol{\sigma}$ lines in all possible ways, so that the diagram is connected.
  - Assign momenta and vector indices to all lines. Contract the vector indices along all continuous lines, not counting dashed lines. Conserve momentum at all vertices.
  - For every contracted $\boldsymbol{\sigma}$ line with momentum $\mathbf{k}$, write $1/(t + Kk^2)$. For every closed loop, multiply by n and integrate $đ\mathbf{k}$.
  - Multiply by the number of ways to get the diagram. For example, a vertex with three $\boldsymbol{\sigma}$ legs gets a factor of 4. We also multiply by the number of contractions. In quantum field theory, this step would instead be 'divide by the symmetry factor', but we haven't put the appropriate $1/n!$ factors in the free energy here.
  - Multiply by $(-1)^{n+1}/n!$ to get the contribution to $\beta H'[\mathbf{m}']$.

- We change variable to $\mathbf{q}' = b\mathbf{q}$ and renormalize the field by z, giving the recursion relations

$$t' = b^{-d}z^2(t + uI(t)), \quad K' = b^{-d-2}z^2K, \quad u' = b^{-3d}z^4u.$$

  As in the Gaussian model, there is a fixed point at $t^* = u^* = 0$, where we set

$$z = b^{1+d/2}$$

  to fix $K' = K$.

We now investigate the RG flow in the vicinity of the Gaussian fixed point.

- Near the Gaussian fixed point, we have the recursion relations

$$t'_b = b^2(t + uI(t)), \quad u'_b = b^{4-d}u.$$

  It is convenient to convert these to continuous flow equations. Letting $b = e^{\ell} \approx 1 + \ell$ and expanding to first order in ℓ, we have

$$\frac{dt}{d\ell} = 2t + \frac{Au\Lambda^d}{t + K\Lambda^2}, \quad \frac{du}{d\ell} = (4-d)u.$$

  To derive the first equation, we used the fact that the integral is over a very thin spherical shell, and collected constants into A. These are differential equations for the RG flow in $\log\Lambda$. The right hand sides are called beta functions in quantum field theory.

- Linearizing about the fixed point, we find

$$\frac{d}{d\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix}2 & A\Lambda^{d-2}/K\\ 0 & 4-d\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}.$$

  The relevance of operators is determined by the eigenvalues of the matrix above; as in the Gaussian model, they are 2 and $4-d$. Nothing has changed from the Gaussian model, except that one of the eigendirections is tilted; thus we haven't learned anything new yet.

- Since the perturbation series is alternating, we expect that at second order, we will have

$$\frac{dt}{d\ell} = 2t + \frac{Au\Lambda^d}{t+K\Lambda^2} - C_1u^2, \quad \frac{du}{d\ell} = (4-d)u - C_2u^2.$$

  In this case, there is now an additional fixed point at $u^* = (4-d)/C_2$ for $d < 4$.

- Since we are treating u as a small parameter, this is only self-consistent if $4-d=\epsilon$ is also small, yielding the 'epsilon expansion'. Below, we will expand to lowest nontrivial order in ϵ, which implies that we need to expand $du/d\ell$ to $O(u^2)$, since $u = O(\epsilon)$, while we only need to expand t to $O(u)$. Hence the value of C_1 doesn't matter.

Note. Wick's theorem. Let variables ϕ_a have Gaussian distribution $p(\phi) = Ae^{-\phi G^{-1}\phi/2}$. We'll use notation for discrete variables here, though in the field theory case they are continuous. Note that

$$\langle e^{B_a\phi_a}\rangle = A\int d\phi\, e^{-\phi G^{-1}\phi/2+B\phi} = A\int d\phi\, e^{-\phi G^{-1}\phi/2}e^{BGB/2} = e^{BGB/2}$$

where we completed the square and shifted. Since the correlator is the Green's function, which is the inverse of the kinetic term, $\langle\phi_a\phi_b\rangle = G_{ab}$. Taylor expanding both sides in B, we have

$$\frac{1}{(2n)!}B_{a_1}\dots B_{a_{2n}}\langle\phi_{a_1}\dots\phi_{a_{2n}}\rangle = \frac{1}{2^n}\frac{1}{n!}B_{a_1}\dots B_{a_{2n}}\langle\phi_{a_1}\phi_{a_2}\rangle\dots\langle\phi_{a_{2n-1}}\phi_{a_{2n}}\rangle.$$

The symmetric tensor $B_{a_1}\dots B_{a_{2n}}$ symmetrizes the correlators ('contractions') on the right-hand side. The number of distinct contractions is $(2n-1)(2n-3)\dots(3)(1)$, which precisely cancels the constants. Then

$$\langle\phi_{a_1}\dots\phi_{a_{2n}}\rangle = \langle\phi_{a_1}\phi_{a_2}\rangle\dots\langle\phi_{a_{2n-1}}\phi_{a_{2n}}\rangle + \text{ other contractions}$$

which correlators of an odd number of fields vanish, just as desired.

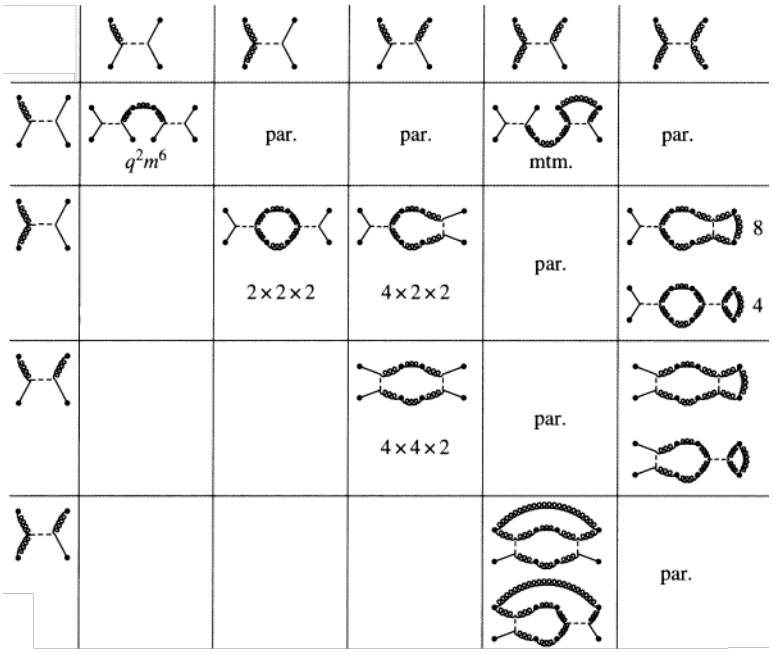
5.3 Wilson–Fisher Fixed Point

Next, we refine our perturbative calculation to second order.

- We now have two factors of U to deal with, so there many types of terms. Note that if one of the vertices has zero $\boldsymbol{\sigma}$ legs, it cannot be connected, so it doesn't contribute to the cumulant. Diagrams with zero $\mathbf{m}$ legs just contribute a constant. We don't include these diagrams in the table above.

- Diagrams with an odd number of $\boldsymbol{\sigma}$ legs must contribute zero; they are eliminated by parity.

- Diagrams with a vertex with three $\boldsymbol{\sigma}$'s and one $\mathbf{m}'$, with two of the $\boldsymbol{\sigma}$'s directly joined together, must be zero by momentum conservation.
- We can group together terms by the power of m they contribute to, given by the number of m legs, giving terms proportional to m^2, m^4, and m^6. The first two contribute to the second-order RG evolution of t and u. The third (from the top-left diagram) contributes to a new term, showing that the space of our current parameters is not closed under RG evolution. We generally get all possible operators allowed by symmetries.
- Note that the top-left diagram is zero when $q \to 0$ by momentum conservation. Since it is analytic, its lowest order contribution is q^2m^6, not m^6. Indeed, we get a whole family of new sixth-order derivative terms.
- The diagrams in the last column contribute to m^2, adding to C_1, which we have argued we can neglect.
- The diagrams in the last row contribute to m^2, but also to q^2m^2, leading to a field renormalization; this determines the critical exponent η. The field renormalization factor will be modified by a factor of $1 + O(u^2)$, which means $dt/d\ell$ is modified at $O(u^2)$ and $du/d\ell$ is modified at $O(u^3)$. Then in both cases we can ignore the effect.



Now we focus on the contribution that counts.

- The terms in the middle of the table contribute to m^4. In particular, we get the product of two propagators, giving a loop integral

$$\int \frac{d\mathbf{k}}{(t + Kk^2)(t + K(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k})^2)}$$

where the $\mathbf{q}_i$ are external momenta. Taylor expanding the integrand gives a renormalization of u, as well as contributions to quartic gradient terms, such as $m^2\nabla^2 m^2$.

- For simplicity, we'll specialize to $n = 1$. Then the three types of diagram contributing to m^2 collapse to one. Expanding to lowest order in the external momenta gives

$$\Delta u = -36u^2 \int_{\Lambda/b}^{\Lambda} \frac{đ\mathbf{k}}{(t + Kk^2)^2}.$$

  The prefactor is because there are $\binom{4}{2}$ ways to choose two σ legs in each vertex, 2 ways to contract the σ's together to form a connected diagram, and a factor of $-1/2$ from the perturbation series.

- Therefore, the RG flow equations are

$$\frac{dt}{d\ell} = 2t + \frac{Au\Lambda^d}{t + K\Lambda^2}, \quad \frac{du}{d\ell} = (4-d)u - \frac{Bu^2\Lambda^d}{(t + K\Lambda^2)^2}, \quad A = \frac{12\Omega_{d-1}}{(2\pi)^d}, \quad B = \frac{36\Omega_{d-1}}{(2\pi)^d}.$$

  In particular, at $d = 4$, u is marginally irrelevant. That is, it does turn out to be irrelevant in the IR, but it decays only logarithmically, in contrast to the power-law decay of t. Marginally irrelevant couplings are relatively common; marginally relevant couplings, which imply asymptotic freedom, are rare.

To make further progress, we work in $d = 4 - \epsilon$.

- As argued earlier, we now have $u = O(\epsilon)$, and we expand $dt/d\ell$ and $du/d\ell$ to leading order in ϵ. We can replace Λ^d with Λ^4, etc., because these will give higher-order corrections. Then

$$\frac{dt}{d\ell} = 2t + \frac{3}{2\pi^2}\frac{\Lambda^4}{t + \Lambda^2}u, \quad \frac{du}{d\ell} = \epsilon u - \frac{9}{2\pi^2}\frac{\Lambda^4}{(t + \Lambda^2)^2}u^2$$

  where we fixed $K = 1$ for simplicity and used $\Omega_4 = 2\pi^2$.

- There is a second fixed point whose location is, to leading order in ϵ,

$$t_* = -\frac{1}{6}\epsilon\Lambda^2, \quad u_* = \frac{2\pi^2}{9}\epsilon.$$

  This is the Wilson–Fisher fixed point. Linearizing around the fixed point, we find

$$\frac{d}{d\ell}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix}2 - \epsilon/3 & 3\Lambda^2/8\pi^2 \\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}$$

  again to leading order in ϵ.

- The eigenvalues at this fixed point are

$$\Delta_t = 2 - \frac{\epsilon}{3}, \quad \Delta_u = -\epsilon.$$

  Only one is positive, so the Wilson–Fisher fixed point can describe the critical behavior. (In a full analysis, we would have to show that all other operators are irrelevant.) Note that u itself doesn't have a well-defined scaling dimension; instead Δ_u means the scaling dimension of the most relevant operator that contains m^4.

- Finally, to extract critical exponents, we note that Δ_t is the scaling dimension of t, so

$$\xi \sim t^{-\nu}, \quad \nu = \frac{1}{\Delta_t} = \frac{1}{2} + \frac{\epsilon}{12}$$

  again to first order in ϵ. Using the hyperscaling relation $\alpha = 2 - d\nu$ gives $\alpha = \epsilon/6$.

- It takes some more work to compute Δ_ϕ, as this is related to the field renormalization we neglected. We find that $\eta = \epsilon^2/6$, so it can be neglected at this order, giving $\Delta_\phi = (d-2)/2 = 1 - \epsilon/2$. Applying scaling results gives all the other critical exponents,

$$\beta = \frac{1}{2} - \frac{\epsilon}{6}, \quad \gamma = 1 + \frac{\epsilon}{6}, \quad \delta = 3 + \epsilon.$$

  Plugging in $\epsilon = 1$ gives a fair approximation to the critical exponents in $d = 3$.

- For general n, we find

$$\frac{dt}{d\ell} = 2t + \frac{n+2}{2\pi^2}\frac{\Lambda^4}{t+\Lambda^2}u, \quad \frac{du}{d\ell} = \epsilon u - \frac{n+8}{2\pi^2}\frac{\Lambda^4}{(t+\Lambda^2)^2}u^2$$

  where we have 5 contributing diagrams; we found the factor of $n+2$ earlier when computing the perturbed propagator.

- Linearizing about the fixed point, we find

$$\Delta_t = 2 - \frac{n+2}{n+8}\epsilon, \quad \Delta_u = -\epsilon$$

  which gives the critical exponents

$$\alpha = \frac{4-n}{2(n+8)}\epsilon, \quad \beta = \frac{1}{2} - \frac{3}{2(n+8)}\epsilon, \quad \gamma = 1 + \frac{n+2}{2(n+8)}\epsilon, \quad \nu^{-1} = 2 - \frac{n+2}{n+8}\epsilon, \quad \delta = 3 + \epsilon$$

  while the field renormalization gives

$$\eta = \frac{n+2}{2(n+8)}\epsilon^2.$$

Note. A summary of our results, for $n = 1$. The first two rows are experimental results; then we list mean field theory (where α is determined from quadratic fluctuations), the critical exponents from hyperscaling assuming $\eta = 0$ and $\nu = 1/2$, and finally the epsilon expansion for $\epsilon = 1$ to linear order in ϵ.

	α	β	γ	δ	η	ν
$d = 2$	0	1/8	7/4	15	1/4	1
$d = 3$	0.1101	0.3264	1.2371	4.7898	0.0363	0.6300
naive MFT	$(4-d)/2$	1/2	1	3	0	1/2
hyperscaling	$(4-d)/2$	$(d-2)/4$	1	$(d+2)/(d-2)$	0	1/2
$d = 4-1$	0.17	0.33	1.17	4	0	0.58

6 Lattice Models

6.1 Introduction

Discrete lattice models are an alternative to statistical field theories which can often be more computationally convenient, especially for series expansions or computer simulation and visualization. By universality, we expect that switching to a lattice model will have no impact on the macroscopic physics. We first introduce the Ising model.

- We place a "spin" $\sigma_i = \pm 1$ at every site of a lattice, which could represent spin up or spin down, or one of two species in a binary mixture, or the presence of absence of a molecule in the lattice gas model.
- We consider only short-range interactions between nearest neighbors $\langle ij \rangle$, giving the Hamiltonian

$$H = \sum_{\langle ij \rangle} \hat{B}(\sigma_i, \sigma_j).$$

- Since $\sigma_i^2 = 1$, the most general decomposition of B is

$$\hat{B}(\sigma, \sigma') = -\hat{g} - \frac{\hat{h}}{z}(\sigma + \sigma') - J\sigma\sigma'.$$

  Here, z is the coordination number (the number of neighbors per site), and we have constrained the second term using the symmetry $B(\sigma, \sigma') = B(\sigma', \sigma)$.

- For N spins, there are 2^N microstates, and the partition function is

$$Z = \sum_{\sigma_i} e^{-\beta H} = \sum_{\sigma_i} \exp\left(K \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i + g \right)$$

  where we have redefined some constants to absorb β.

- We see that g is just an overall constant, and h could represent a uniform external field. Zooming out, we expect that the Ising model becomes the $n = 1$ Landau–Ginzburg Hamiltonian. For $h = 0$, there is a phase transition as K (equivalently T) is varied.

The Ising model has several generalizations.

- The $O(n)$ model. Every lattice site is now occupied by an n-component unit vector $\mathbf{S}_i$. One possible nearest-neighbor interaction is

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \hat{\mathbf{h}} \cdot \sum_i \mathbf{S}_i.$$

  More generally, the interaction term could be any function of $\mathbf{S}_i \cdot \mathbf{S}_j$.

- The $O(n)$ model reduces to the Ising model, the XY model, and the classical Heisenberg model, for $n = 1$, $n = 2$, and $n = 3$ respectively. Coarse-graining it gives the Landau–Ginzburg Hamiltonian for general n, but note that this field does not have a unit length constraint, because averaging unit vectors doesn't necessarily yield a unit vector. If we want to include the unit length constraint explicitly, we get a nonlinear σ model.

- The Potts model. Each lattice site is occupied by a q-valued spin $S_i \in \{1, \dots, q\}$, and the interactions between the spins are described by

$$H = -J \sum_{\langle ij \rangle} \delta_{S_i, S_j} - \hat{h} \sum_i \delta_{S_i, 1}.$$

  The field h now breaks an S_q permutation symmetry. The Ising model is the case $q = 2$, since $S_2 = \mathbb{Z}_2$. For higher q, very different behavior occurs; for instance the transitions for $q > 3$ in $d = 3$ are discontinuous.

- As a physical example, consider the absorption of Krypton onto graphene. The Krypton atoms are larger than the lattice spacing, so they forbid adjacent hexagons from being filled. This can be represented using a Potts model.

There are several ways to analyze a lattice system.

- Exact solution. This is only possible for a few systems, such as the 1D and 2D Ising models.
- Perturbative (series) expansions. There are separate expansions for high T (perturbative interactions) and low T (low-energy excitations), expanding about a disordered or ordered state. Critical behavior can be extracted from the singularities of the series.
- Numerical simulation by Monte Carlo, such as the Metropolis algorithm. This is easier for lattice systems than statistical fields.
- Real space RG. We perform coarse graining by averaging neighboring lattice sites; we will use this method below. It is somewhat difficult when new interactions are generated by the RG, as we don't have a small parameter to expand in; we will introduce some approximations below, but they are uncontrolled.

6.2 Ising Model in $d = 1$

The Ising model in $d = 1$ is simple to solve. First, we use transfer matrices.

- We note the partition function can be written as

$$Z = \sum_{\sigma_i} \exp\left(K \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i \right) = \sum_{\sigma_i} \prod_{j=1}^{N} T_{\sigma_j, \sigma_{j+1}}, \quad T_{ij} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

  where we have assumed the spins are periodic, $\sigma_{N+1} = \sigma_N$. Here T is called the transfer matrix, and the partition function has the form

$$Z = \operatorname{tr} T^N.$$

- The eigenvalues of the transfer matrix are

$$\lambda_\pm = e^K \cosh(h) \pm \sqrt{e^{2J} \sinh^2(h) + e^{-2J}}.$$

  The partition function is thus

$$Z = \lambda_+^N + \lambda_-^N \approx \lambda_+^N$$

  for large N. Given this we can easily compute the magnetization $m(h)$ by differentiation.

- The transfer matrix can also be used to compute averages of observables. For instance,

$$\langle \sigma_j \rangle = \frac{1}{Z} \sum_{\sigma_i} \sigma_j e^{-\beta H} = \frac{1}{Z} \operatorname{tr}(T^{j-1} \sigma^z T^{N-j+1}) = \frac{1}{Z} \operatorname{tr}(T^N \sigma^z)$$

  where we used $T_{\sigma_j, \sigma_{j+1}} \sigma_j = (T\sigma^z)_{\sigma_{j-1}\sigma_j}$. To get an explicit result, we must expand σ^z in the eigenbasis of T. After some messy calculation, we find

$$\lim_{N\to\infty} \langle \sigma_j \rangle = \frac{\alpha^2 - 1}{\alpha^2 + 1}, \quad \alpha = e^{2K} \sinh(h) + \sqrt{1 + e^{4J} \sinh^2(h)}.$$

- One useful application of this result is that

$$\lim_{h\to 0} \lim_{N\to\infty} \langle \sigma_j \rangle = 0$$

  for any nonzero temperature, which establishes that spontaneous symmetry breaking does not happen in the 1D Ising model.

- We may also compute correlators, such as

$$\langle \sigma_j \sigma_{j+r} \rangle = \frac{1}{Z} \operatorname{tr}(\sigma^z T^r \sigma^z T^{N-r}).$$

  In particular, in the case of zero external field,

$$\langle \sigma_j \sigma_{j+r} \rangle|_{h=0} = \frac{\lambda_+^{N-r} \lambda_-^r + \lambda_-^{N-r} \lambda_+^r}{\lambda_+^N + \lambda_-^N} \approx \left(\frac{\lambda_-}{\lambda_+} \right)^r = e^{-r/\xi}, \quad \xi = \frac{1}{\log \coth(\beta J)}.$$

We now perform an exact RG analysis of the Ising model in $d = 1$.

- Our goal is to reduce the number of degrees of freedom by a factor b, while preserving the partition function

$$Z = \sum_{\sigma_i}^{N} e^{-\beta H[\sigma_i]} = \sum_{\sigma'_{i'}}^{N/b} e^{-\beta H'[\sigma'_{i'}]}.$$

- There are many ways to construct the new $\sigma'_{i'}$. For $b = 2$, we could use the average

$$\sigma'_i = (\sigma_{2i-1} + \sigma_{2i})/2.$$

  However, this is inconvenient as σ'_i now has three possible values instead of two. A simpler choice is to just take one of the spins,

$$\sigma'_i = \sigma_{2i-1}.$$

  This is called a 'decimation', as it just removes the even spins.

- There is no need to rescale the new spins by ζ, as they continue to take the values ± 1.

- To implement the decimation, write the even spins as s_i. Then we just 'integrate them out' (i.e. sum over them in the partition function), for

$$Z = \sum_{\sigma_i}^{N} \exp\left(\sum_{i=1}^{N} B_{\sigma_i, \sigma_{i+1}} \right) = \sum_{\sigma'_i}^{N/2} \sum_{s_i}^{N/2} \exp\left(\sum_{i=1}^{N/2} B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1}) \right).$$

- Performing the summation, we find the renormalized interaction B' must satisfy

$$\exp\left(B'(\sigma_i', \sigma_{i+1}')\right) = \sum_{s_i=\pm 1} \exp\left(B(\sigma_i', s_i) + B(s_i, \sigma_{i+1}')\right).$$

  Another way to think about this is that, if we define the transfer matrix $\langle \sigma_i | T | \sigma_{i+1} \rangle$, the new transfer function is $T' = T^2$.

- Defining

$$B(\sigma_1, \sigma_2) = g + \frac{h}{2}(\sigma_1 + \sigma_2) + K\sigma_1\sigma_2, \quad B'(\sigma_1', \sigma_2') = g' + \frac{h'}{2}(\sigma_1' + \sigma_2') + K'\sigma_1'\sigma_2'$$

  we can straightforwardly find the RG recursion relations

$$g' = 2g + \delta g(K, h), \quad h' = h + \delta h(K, h), \quad K' = K'(K, h)$$

  where δg and δh are somewhat complicated functions. Since g is just an additive constant, it doesn't affect probabilities and hence doesn't appear in the RG recursion relations.

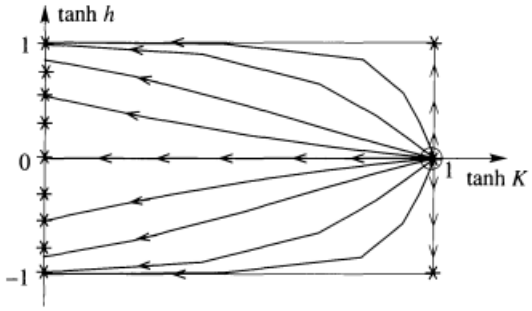
We now analyze the behavior of the RG flow.

- The $h = 0$ subspace is closed by symmetry. Fixing $h = 0$, we find

$$e^{2K'} = \frac{e^{2K} + e^{-2K}}{2} \quad \rightarrow \quad K' = \frac{1}{2}\log\cosh 2K.$$

  We see there is a stable infinite temperature fixed point at $K^* = 0$, which represents perfect disorder/independent spins. There is also an unstable zero temperature fixed point at $K^* \to \infty$, describing the ordered phase. Since no finite value of K goes to this fixed point, the Ising model doesn't have a phase transition in $d = 1$, as expected.

- Adding a finite h back in, we see that the flow comes out of the $K, h = \infty, 0$ fixed point, ending at $K = 0$ and h finite. So any generic point ends up at $K = 0$ and some arbitrary h, describing independent spins as before.



- Linearizing the recursion relations about the $K \to \infty$ fixed point yields

$$e^{-K'} = \sqrt{2}e^{-K}, \quad h' = 2h.$$

  We can then use these to extract scaling behavior near this fixed point.

- Using the fact that $\xi' = \xi/2$ (since we decimate half the spins), we find

$$\xi(e^{-K}, h) = e^{2K} g_\xi(he^{2K})$$

by the usual method. However, the divergence of ξ as $T \to 0$ is ambiguous, since it depends on how we define $T(K)$, so we can't get the critical exponent ν.

- Using the hyperscaling assumption $f_{\text{sing}} \sim \xi^{-1}$, we find

$$f_{\text{sing}} = e^{-2K} g_f(he^{2K}).$$

The zero-field susceptibility behaves as

$$\chi(K) \sim \left.\frac{\partial^2 f}{\partial h^2}\right|_{h=0} \sim e^{2K}$$

which is the same form as the behavior of the correlation length. Since we have already shown $\chi \sim \xi^{2-\eta}$, we conclude $\eta = 1$.

6.3 Ising Model in $d = 2$

In $d > 1$, the transfer matrix formalism is less useful but still has some benefit.

- For an $N \times N$ square lattice with periodic boundary conditions, we let $\boldsymbol{\sigma}_i$ be a vector of the N spins in column i. Then the partition function for zero external field can be written as

$$Z = \operatorname{tr} T^N, \quad \langle \boldsymbol{\sigma}_k | T | \boldsymbol{\sigma}_{k+1} \rangle = \exp\left(K \sum_{j=1}^{N} \sigma_{k,j}\sigma_{k+1,j} + \frac{1}{2}(\sigma_{k,j}\sigma_{k,j+1} + \sigma_{k+1,j}\sigma_{k+1,j+1}) \right).$$

The transfer matrices are $2^N \times 2^N$.

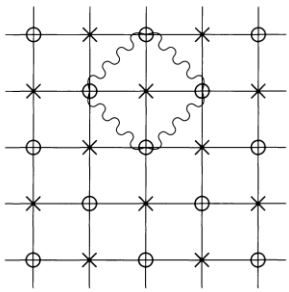
- As before, in the thermodynamic limit we have

$$f = \lim_{N\to\infty} \frac{F}{N^2} = -\lim_{N\to\infty} \frac{k_B T}{N} \log \lambda_{\max}.$$

Hence the thermodynamic properties can be found by looking at only a $2^N \times 2^N$ matrix, though the partition function is a sum of 2^{N^2} terms. Still, this is analytically intractable.

In $d > 1$ the decimation procedure can't be performed exactly, because new interactions will appear. Hence we must use approximations in the RG.

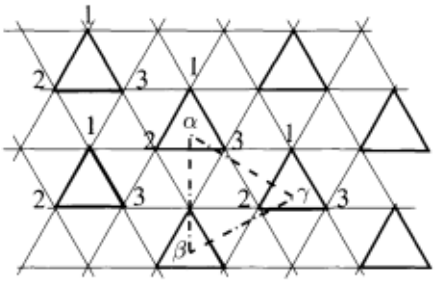
Example. Suppose we separate a square lattice into two square sublattices, with $b = \sqrt{2}$.



Since all four remaining spins shown above are symmetric, we will get new interactions between the left and right spins, which are next-to-nearest neighbor interactions in the new sublattice! Moreover, we get a four-point interaction between all four of the spins. Further RG steps only make the problem more complex.

As an extended example, we consider the NvL cumulant approximation.

- We work in a triangular lattice divided into three-spin triangular cells.



  The renormalized spin for the cell α is determined by the majority rule,

$$\sigma'_\alpha = \text{sign}(\sigma^1_\alpha + \sigma^2_\alpha + \sigma^3_\alpha).$$

  The exact renormalized interactions are obtained from

$$e^{-\beta H'[\sigma'_\alpha]} = \sum_{\sigma^i_\alpha} e^{-\beta H[\sigma^i_\alpha]}$$

  where we sum over spin configurations $\{\sigma_\alpha\}$ that map to the desired spin configuration $\{\sigma'_\alpha\}$ under the majority rule.

- We split the Hamiltonian as

$$\beta H = \beta H_0 + U, \quad -\beta H_0 = K \sum_{\alpha,i} \sigma^i_\alpha \sigma^{i+1}_\alpha$$

  where for simplicity we have set $h = g = 0$. The 'free' Hamiltonian H_0 only contains intracell interactions, which are trivial to deal with, and the 'perturbation' U contains interactions between cells,

$$-U = K \sum_{\langle\alpha,\beta\rangle} \sigma^i_\beta \sigma^j_\alpha + \sigma^k_\beta \sigma^l_\alpha.$$

  The sum is over nearest neighbor triangular cells, and the indices i, j, k, and l depend on the relative orientation of α and β.

- We now expand perturbatively in U. This is an uncontrolled expansion, since there is no sense in which U is smaller compared to βH_0.

- Expanding in a power series anyway, we have

$$e^{-\beta H'[\sigma'_\alpha]} = \sum_{\sigma^i_\alpha} e^{-\beta H_0[\sigma^i_\alpha]} \left(1 - U + \frac{U^2}{2} - \cdots\right).$$

- Taking the logarithm, we conclude that

$$-\beta H'[\sigma'_\alpha] = -\log Z_0[\sigma'_\alpha] + \langle U\rangle_0 - \frac{1}{2}\left(\langle U^2\rangle_0 - \langle U\rangle_0^2\right) + \dots.$$

  We get a series of cumulants, where the expectation values are taken with respect to βH_0, with the same restriction of fixed σ'_α.

- For a fixed σ'_α, there are 4 spin configurations per α that contribute on the right-hand side. For example, the partition function is

$$\left(e^{3K} + 3e^{-K}\right)^{N/3}.$$

  Since this is independent of σ'_α, it simply contributes an additive constant, which we ignore.

- We get nontrivial results from the first cumulant,

$$-\langle U\rangle_0 = 2K \sum_{\langle\alpha,\beta\rangle} \langle\sigma^i_\alpha\rangle_0 \langle\sigma^j_\beta\rangle_0.$$

  Here, we've split the two-point correlation function since spins on different triangular cells are independent under βH_0. Since each spin is symmetric, the indices i and j are arbitrary.

- If the sum were not restricted, we would have $\langle\sigma^i_\alpha\rangle_0 = 0$ always. However, we get a nonzero answer here because we are conditioning on the value of the majority σ'_α. Casework gives

$$\langle\sigma^i_\alpha\rangle_0 = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\sigma'_\alpha.$$

- Substituting back into the Hamiltonian, we get the nearest-neighbor interaction

$$-\beta H'[\sigma'_\alpha] = 2K\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)^2 \sum_{\langle\alpha,\beta\rangle} \sigma'_\alpha\sigma'_\beta + \text{const} + O(U^2).$$

  If we had gone to higher-order terms, we would have more interactions.

We now analyze the RG flow.

- We have shown above that at lowest order,

$$K' = 2K\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right)^2.$$

- As before, $K = 0$ and $K = \infty$ are both fixed points, but now both are stable! As a result, by continuity, there must be an unstable fixed point in between, which numerically satisfies

$$K^* \approx 0.3356.$$

  This is the location of the critical point. The known exact value is 0.2747.

- By linearizing about the fixed point, we find that deviations are multiplied by 1.624 on each iteration. Distances are shrunk by $b = \sqrt{3}$ each time, so the thermal eigenvalue is

$$y_t \approx \frac{\log 1.624}{\log \sqrt{3}} \approx 0.883$$

  while the exact value is $y_t = 1$. Note that we don't have the temperature ambiguity seen in the 1D Ising model, as we're now expanding about a point with finite K. We can then use this result to find other critical exponents.

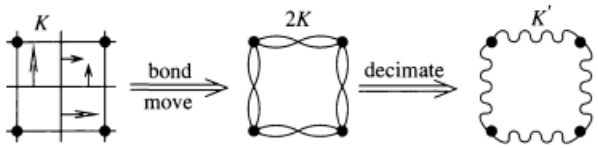
- Similarly, we can introduce a finite external field h as a second perturbation. By similar arguments, we find

$$h' = 3h\left(\frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}\right) \quad \rightarrow \quad y_h \approx 1.37.$$

- In general we find reasonable agreement with the exact values, somewhat better than mean field theory. We can get further improvement by going to the next order.

Next, we consider the Migdal-Kadanoff bond moving approximation.

- Consider performing $b = 2$ RG for the Ising model on a square lattice, as shown below.



  The complications come from the bonds in the middle of the square. However, if we simply ignore them, then the recursion relation for K is just the same as the 1D case, so the approximation is not good; we have weakened the interactions too much. A better approximation is to imagine 'moving' the inside bonds to the outside, effectively doubling the bond strength K.

- The above gives the slightly modified recursion relation

$$K' = \frac{1}{2}\log\cosh 4K.$$

  We find low-temperature and high-temperature stable fixed points, along with a third fixed point at $K^* \approx 0.305$. About this fixed point, $y_t \approx 0.747$.

- Similarly, in dimension d, the approximation gives

$$K' = \frac{1}{2}\log\cosh 2 \times 2^{d-1}K.$$

  Generally, the results become quantitatively less accurate for higher d.

- The bond-moving approximation complements the mean-field approach, as it works best in lower dimensions: it gets the case $d = 1$ correct exactly, and correctly identifies the lower critical dimension $d_l = 2$.

- More generally, in any one dimensional system, the transfer matrix is raised to the b power. Then in d dimensions, the Migdal-Kadanoff approach consists of replacing K with $b^{d-1}K$, then raising the transfer matrix to the b power.

Note. The Peierls argument can be used to extract qualitative information about lattice models; it is a rough version of the series expansions we perform below.

Consider a 2D Ising model with N total spins with the boundary constrained to point up. The system may contain domain walls of length L, across which the spin flips. The number $m(L)$ of such domain walls is bounded by

$$m(L) \leq (4N)3^{L-1}.$$

Each domain wall of length L may contribute at most $L^2/16$ down spins, where equality is achieved if the domain wall is square and is not inside another domain wall. Finally, the energy cost of a domain wall is precisely $2LK$, so the probability of a specific domain wall appearing is at most e^{-2LK}. Putting everything together, we have

$$\langle N_- \rangle \leq 4N \sum_L \frac{L^2}{16} 3^{L-1} e^{-2LK} = \frac{N}{12} \sum_L L^2 (3e^{-LK})^L \approx 108 N e^{-LK}$$

in the low temperature limit. Hence in this limit $\langle N_- \rangle \ll N$, so there can indeed exist an ordered phase. A similar argument in 1D, which we've seen above, instead shows that domain walls proliferate because the energy cost of a domain wall doesn't grow with L.

6.4 Series Expansions

Lattice models can also be studied by series expansions of the partition function Z. In this section, we consider series expansions for the Ising model, $-\beta H = K \sum_{\langle ij \rangle} \sigma_i \sigma_j$ in d dimensions and N sites, where $K = \beta J > 0$. First we consider the low-temperature limit of the Ising model.

- The ground states have all the spins facing the same direction, with a weight $2e^{NdK}$ in the partition function.
- Every bond that is broken comes with an energetic cost of $2K$, and hence a probability cost of $\alpha = e^{-2K}$ relative to the ground states. We thus perform a series expansion in α.
  - The first term comes from a single flipped spin, with cost α^{2d} and multiplicity N.
  - The next term comes from two adjacent flipped spins, with cost α^{4d-2} and multiplicity dN.
  - The next term comes from two non-adjacent flipped spins, with cost α^{4d} and multiplicity $N(N-1-2d)/2$.

  In general, the expansion has the form

$$Z = 2e^{NdK} \sum_{\text{droplets}} e^{-2K \times \text{boundary of droplet}}.$$

- We expect the free energy to be extensive since the forces are short-ranged, so that $\log Z$ only has terms proportional to N. Therefore there must be some cancellation of the higher N terms when we take the log.
- Intuitively, taking the log takes us from general diagrams to connected diagrams. A connected diagram always comes with a factor of N, corresponding to where its center is, while diagrams with m components generically have factors of N^m. The correspondence here isn't exact because there are some annoying factors coming from the fact that space is discrete (so we get $N(N-1-2d)/2$ above instead of $N^2/2$).

- Differentiating, we can extract a series for the specific heat, $C = \sum \alpha_\ell u^\ell$ where u is our expansion variable. Given the expected divergence of the heat capacity, we expect that the series looks like
$$C \sim \left(1 - \frac{u}{u_c}\right)^{-\alpha} = 1 + \frac{\alpha}{u_c}u + \frac{\alpha(\alpha+1)}{2!u_c^2}u^2 + \cdots$$
near the divergence. Then we can extract u_c and α from the ratio of subsequent terms,
$$\frac{a_\ell}{a_{\ell-1}} = u_c^{-1}\left(1 + \frac{\alpha - 1}{\ell}\right).$$

- Formally, there's no reason this is guaranteed to work, as we can always add analytic terms to C without changing the divergence. However, it works well in practice.

- Another problem is that we expect a factor of the form $(1 - u/u_*)^\beta$ from every singularity u_* of the specific heat. While there are no more singularities on the real axis, there are some on the complex plane in our expansion in $u = \alpha$ which cause the series to alternate in sign. To remove this problem, we can change parameters so that the complex poles are pushed away; in practice $u = \tanh K$ is used.

We can also construct an expansion in the limit of high temperature.

- In this limit, the spins are nearly independent, so we expand in powers of β. We have
$$Z = \operatorname{tr} e^{-\beta H} = \operatorname{tr}\left(1 - \beta H + \frac{\beta^2 H^2}{2} - \cdots\right).$$
Each of the terms of the series can be thought of as an expectation value, taken with respect to a trivial Hamiltonian $e^{-\beta H_0}$ where $H_0 = 0$, describing independent spins. Then
$$Z = Z_0\left(1 - \beta\langle H\rangle_0 + \frac{\beta^2}{2}\langle H^2\rangle_0 + \cdots\right)$$
where the expectation values are taken with respect to H_0.

- Taking the log gives us cumulants as usual,
$$\log Z = \log Z_0 - \beta\langle H\rangle_0 + \frac{\beta^2}{2}(\langle H^2\rangle_0 - \langle H\rangle_0^2) - \cdots.$$

- For the Ising model, a more convenient method is to note that
$$e^{K\sigma_i\sigma_j} = \frac{e^K + e^{-K}}{2} + \frac{e^K - e^{-K}}{2}\sigma_i\sigma_j = \cosh K(1 + t\sigma_i\sigma_j), \quad t = \tanh K.$$
Then the partition function can be written as
$$Z = (\cosh K)^{Nd}\sum_{\{\sigma_i\}}\sum_{\langle ij\rangle}(1 + t\sigma_i\sigma_j).$$
This allows a systematic expansion in t. Geometrically, this is nicer because each 'diagram' can be specified by a subset of the bonds, while in our original expansion (in β) the $\langle H\rangle^n$ terms would include diagrams where each bond could be included up to n times.

- A bond between sites i and j contributes a factor of $\sigma_i\sigma_j$, but since spins are independent, generic contributions just average to zero. We can only get a nonzero result if every spin σ_i appears raised to an even power. Diagrammatically, this happens if the bonds drawn form collections of closed paths on the lattice.

- Performing the $\{\sigma_i\}$ summation, we conclude

$$Z = 2^N(\cosh K)^{Nd} \sum_{\text{closed graphs}} t^{\text{bonds in graph}}.$$

  Note that the low-temperature expansion is over vertices, while the high-temperature expansion is over bonds.

Example. 1D Ising model with N sites on an open chain. There are no closed graphs, so

$$Z = 2^N \cosh K^{N-1}$$

which is the exact result. Similarly, we may calculate the correlation function

$$\langle\sigma_m\sigma_n\rangle = \frac{(\cosh K)^{N-1}}{Z} \sum_{\{\sigma_i\}} \sum_{\langle ij\rangle} (1 + t\sigma_i\sigma_j)\sigma_m\sigma_n.$$

Diagrammatically, the graphs that contribute are 'closed graphs' with endpoints on the 'external sources' m and n. The single graph is a chain running from m to n, so

$$\langle\sigma_m\sigma_n\rangle = t^{|m-n|} \quad \to \quad \xi = -\frac{1}{\log\tanh K}.$$

This is in agreement with our previous RG results.

Example. 1D Ising model with N sites on a closed chain. There is now a single closed graph,

$$Z = (2\cosh K)^N(1 + t^n) = 2^n(\cosh^N K + \sinh^N K).$$

Taking the log, we see that the free energy approaches that of the open chain in the thermodynamic limit, as the contribution from the single closed graph is not extensive. By similar reasoning, the correlation function is now

$$\langle\sigma_m\sigma_n\rangle = \frac{t^{|m-n|} + t^{N-|m-n|}}{1 + t^N}$$

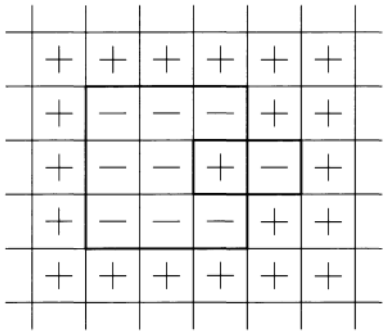
which is appropriately symmetric between m and n.

6.5 Duality in the Ising Model

There is a hidden symmetry relating the low-temperature and high-temperature expansions for the Ising model on the square lattice.

- Given a planar lattice, one can construct a 'geometrical dual' by constructing dual lattice points at the center of each original plaquette, then drawing an edge between dual points for each neighboring plaquette. This swaps vertices and edges.

- Doing this for the Ising model gives an exact correspondence between islands and closed graphs! One example is shown below.



  The summations in the two series are identical, except that e^{-2K} is replaced with $\tanh K$.

- Taking the log, we conclude that

$$\frac{\log Z}{N} = 2K + g(e^{-2K}) = \log 2 + 2\log\cosh K + g(\tanh K).$$

  The arguments of g above are related by the duality condition

$$D(K) = -\frac{1}{2}\log\tanh K.$$

  One can show that $D(D(K)) = K$, so D exchanges pairs of low K and high K.

- Since $g(K) = g(D(K))$+analytic, if $g(K)$ is singular at K_c, it is also singular at $D(K_c)$. However, since we expect that the free energy is analytic everywhere except at the phase transition, the critical point must be self-dual. Solving yields

$$K_c = \frac{1}{2}\log(1+\sqrt{2}) \approx 0.441.$$

- Similarly, we can find duals for the Potts model and the XY model. Duality provides constraints on the shape of phase boundaries, but generally no information on critical exponents.

- Not all planar lattices are self-dual: the dual of a triangular lattice is a hexagonal lattice. In the case of self-duality, we can find the exact value of K_c.

Next, we turn to the 3D Ising model on a cubic lattice.

- The low-temperature expansion has the form

$$Z = 2e^{3NK}\sum_{\text{droplets}} e^{-2K\times\text{area of boundary}}$$

  while the high-temperature expansion has the form

$$Z = 2^N\cosh K^{3N}\sum_{\text{closed graphs}} t^{\text{bonds in graph}}.$$

  Droplets and closed graphs don't correspond in 3D, so the 3D Ising model is not self-dual.

- To construct the dual, we start with the low-temperature expansion. Since it depends on the area of a boundary of a droplet, the high-temperature series of the dual model sums over closed sets of plaquettes. Therefore the plaquettes correspond to the bonds of the dual model.
- The objects connecting the plaquettes together are the bonds of the original model; hence the spins $\widetilde{\sigma}_i = \pm 1$ of the dual model sit on the bonds of the original model.
- Since we know the dual model's partition function is

$$Z \sim \sum_{\{\widetilde{\sigma}_P^i\}} \prod_P (1 + \tanh \widetilde{K} \widetilde{\sigma}_P^1 \widetilde{\sigma}_P^2 \widetilde{\sigma}_P^3 \widetilde{\sigma}_P^4)$$

  we can work backwards to find the Hamiltonian,

$$-\beta H = \widetilde{K} \sum_{\text{plaquettes } P} \prod \widetilde{\sigma}_P^i.$$

- The dual model is called a $\mathbb{Z}_2$ lattice gauge theory, because it has a local symmetry: if we flip all spins on edges connected to a single site, the energy is unchanged. Since it is dual to the 3D Ising model, it must have a phase transition.

Note. Elitzur's theorem states that spontaneous symmetry breaking (SSB) is impossible in models with a local symmetry. Specifically, it states that the expectation value of any local, non-gauge invariant observable must vanish. In particular, the model above can't have a magnetized phase.

To understand the theorem, consider the case of the 2D Ising model. To see SSB, we consider adding a small symmetry-breaking field $h/2$, then taking the thermodynamic limit $N \to \infty$, then finally taking $h \to 0$. One way to prevent SSB is to transition from the ground state to the state with all spins flipped; this costs Nh energy, which becomes infinite in the thermodynamic limit, so it is impossible. (SSB can still be prevented by local spin flips, as in the 1D Ising model; here we're just saying the global $\mathbb{Z}_2$ symmetry plays no role.)

However, in the case of the lattice gauge theory, we can flip any spin by performing a local gauge transformation, which only costs energy $6h$. This goes to zero in the appropriate limits, so SSB is impossible.

Now we turn to Wilson loops and confinement.

- Given the above, the phase transition of the $\mathbb{Z}_2$ lattice gauge theory must have a nonlocal, gauge-invariant order parameter. This turns out to be the Wilson loop,

$$C_S = \left\langle \prod_{i \in S} \widetilde{\sigma}_i \right\rangle$$

  where S is a closed loop; it is gauge-invariant because gauge transformations flip spins in pairs. We can think of C_S as simply a complicated correlation function. We can then detect a phase transition by seeing how C_S depends on the size of S.
- In the high-temperature limit, we have

$$C_S \sim \sum_{\{\widetilde{\sigma}_i\}} \prod_{i \in S} \widetilde{\sigma}_i \prod_P (1 + \tanh \widetilde{K} \widetilde{\sigma}_P^1 \widetilde{\sigma}_P^2 \widetilde{\sigma}_P^3 \widetilde{\sigma}_P^4).$$

The terms in the expansion are surfaces, and the terms that contribute are the surfaces with S as boundary, weighted by their area. Therefore

$$C_S \sim (\tanh \widetilde{K})^{\text{area of } S} \sim \exp\left(-f(\tanh \widetilde{K}) \times \text{area of } S\right)$$

where we consider only the lowest-order term, i.e. the surface with minimal area.

- In the low-temperature limit, we expand over flipped bonds. The zeroth order term has $C_S = 1$, while there are P_S first order terms (with weight $e^{-8\widetilde{K}}$) with $C_S = -1$. Then

$$C_S \sim 1 - 2P_S e^{-8\widetilde{K}} \sim \exp\left(-2e^{-8\widetilde{K}} P_S\right)$$

where P_S is the perimeter of S.

- We see that the high and low-temperature limits have qualitatively different decays for the Wilson loop, depending on the area and perimeter respectively. Therefore they are separated by a phase transition.

- Lattice gauge theory can serve as an analog of QED; we can add a 'matter' field by placing spins on the sites. Adjacent sites are then connected by a gauge field spin. A gauge transformation consists of flipping a site and all adjacent edges.

- Regarding one of the lattice directions as time, consider a Wilson loop with time t and distance x. It corresponds to the statistical weight for creating two particles separated by distance x and propagating them for time t, which is $e^{-U(x)t}$.

- In the high-temperature phase, we have $U(x)t \sim |x|t$, so $U(x) \sim x$ and the particles are confined together. In the low-temperature phase, we instead find $U(x) \sim (|x| + t)/t$, which goes to a constant at large distances. Then the particles are asymptotically free.

6.6 Summing over Phantom Loops

In this section, we approximately sum the high-temperature series for the Ising model.

- The high temperature series for the Ising model on a d-dimensional hypercubic lattice is

$$Z = 2^N \cosh^{dN} K \times S, \quad S = \sum_{\text{closed graphs}} t^\ell$$

where ℓ is the total perimeter. A closed graph can be always decomposed into a sum of closed loops, where each loop doesn't intersect itself or any other loop.

- Now consider the quantity S' so that

$$\Xi = \log S' = \sum_{\text{closed loops}} t^\ell.$$

That is, Ξ is the sum over connected contributions to S. Usually, we would have $S' = S$. However, S' counts closed graphs containing loops that overlap each other, i.e. edges can be counted multiple times. We will neglect these in our approximation.

- We will further approximate by allowing edges to be counted multiple times in Ξ. Let

$$\Xi' = \sum_{\text{closed random walks}} t^\ell$$

and approximate $S \approx e^{\Xi'}$. Here, ℓ is the length of the walk. The true S is a gas of self-avoiding loops, while $e^{\Xi'}$ contains 'phantom loops' which pass through each other and themselves.

- Given a random walk, we can get about N identical random walks by translation (ignoring boundary effects). Moreover, every random walk is counted ℓ times, since the starting point in the walk is arbitrary. Then

$$\Xi' = N \sum_\ell \frac{t^\ell}{\ell} (\text{number of closed walks of } \ell \text{ steps starting and ending at } \mathbf{0})$$

- Define the matrix

$$\langle \mathbf{i}|W(\ell)|\mathbf{j}\rangle = \text{number of walks from } \mathbf{j} \to \mathbf{i} \text{ in } \ell \text{ steps.}$$

Then we have

$$\frac{\Xi'}{N} = \frac{1}{2} \sum_\ell \frac{t^\ell}{\ell} \langle \mathbf{0}|W(\ell)|\mathbf{0}\rangle.$$

The factor of 2 is because the same loop can be traversed in two directions.

- Next, consider the correlation function $\langle \sigma(\mathbf{0})\sigma(\mathbf{r})\rangle$. To compute this, we sum over all graphs consisting of a closed graph and a path from $\mathbf{0}$ to $\mathbf{r}$, and divide by the partition function (which just counts closed graphs). However, in the phantom loop approximation, the path from $\mathbf{0}$ to $\mathbf{r}$ doesn't affect anything else; the partition function cancels, giving

$$\langle \sigma(\mathbf{0})\sigma(\mathbf{r})\rangle \approx \sum_\ell t^\ell \langle \mathbf{r}|W(\ell)|\mathbf{0}\rangle.$$

Given this setup, we now investigate the properties of $W(\ell)$ and the correlation function.

- Since the random walks are Markovian, they satisfy a simple recursion relation; in matrix form, this is just the statement $W(\ell) = T^\ell$ where $T = W(1)$ is the transfer/adjacency matrix.

- Because of the translational symmetry of the lattice, T is diagonalized in Fourier space, with

$$T|\mathbf{q}\rangle = \lambda_{\mathbf{q}}|\mathbf{q}\rangle, \quad \lambda_{\mathbf{q}} = 2\sum_\alpha \cos q_\alpha.$$

- The correlation function is thus

$$\langle \sigma(\mathbf{0})\sigma(\mathbf{r})\rangle \approx \sum_\ell \langle \mathbf{r}|(tT)^\ell|\mathbf{0}\rangle = \langle \mathbf{r}|(1-tT)^{-1}|\mathbf{0}\rangle = \sum_{\mathbf{q}} \langle \mathbf{r}|\mathbf{q}\rangle \frac{1}{1-t\lambda_{\mathbf{q}}} \langle \mathbf{q}|\mathbf{0}\rangle.$$

Replacing the sum with an integral, we conclude

$$\langle \sigma(\mathbf{0})\sigma(\mathbf{r})\rangle \approx \int \bar{d}^d\mathbf{q} \frac{e^{i\mathbf{q}\mathbf{r}}}{1 - 2t\sum_\alpha \cos q_\alpha}.$$

- For small t (i.e. high temperatures), the correlation function is dominated by the shortest path, $\langle\sigma(\mathbf{0})\sigma(\mathbf{r})\rangle \sim t^r$, giving an exponential decay. We can think of the path as a string connecting the two points with a very high tension.
- As t increases, the 'string tension' decreases, and longer paths can contribute. Longer paths have less weight, but there are more of them. In our approximation, the number of paths goes roughly as $(2d)^\ell$, while the weight goes as t^ℓ. Then there is a singularity at $2dt_c = 1$, where arbitrarily long paths become important; this is the Ising ordering phase transition.
- Near the transition, the correlation function will be dominated by long paths, i.e. low momentum components. We thus expand the denominator at low q,

$$1 - 2t\sum_\alpha \cos q_\alpha = (1-2dt) + tq^2 + O(q^4) \approx t_c(\xi^{-2} + q^2 + O(q^4)), \quad \xi = \left(\frac{1-2dt}{t_c}\right)^{-1/2}.$$

  The correlation function is the Fourier transform of $1/(q^2+\xi^{-2})$, exactly as we found through the Landau–Ginzburg approach.
- The correlation length diverges as $\xi \sim (t_c - t)^{-1/2}$, giving the Gaussian exponent $\nu = 1/2$. Note that here, t is not the reduced temperature parameter, it is $\tanh K$. In particular, t increases when the temperature decreases! However, this doesn't matter for the critical exponent because near the critical point, t and the reduced temperature are linearly related.

We can also use our machinery to investigate the free energy.

- The singular part of the free energy is

$$f_{\text{sing.}} = \frac{\log S}{N} \approx \frac{\Xi'}{N} = \frac{1}{2}\sum_\ell \langle\mathbf{0}|\frac{t^\ell T^\ell}{\ell}|\mathbf{0}\rangle = -\frac{1}{2}\langle\mathbf{0}|\log(1-tT)|\mathbf{0}\rangle.$$

  Switching to Fourier space as above, we conclude

$$f_{\text{sing.}} \approx -\frac{1}{2}\int đ\mathbf{q}\, \log\left(1 - 2t\sum_\alpha \cos q_\alpha\right).$$

- Near the critical point, the argument of the logarithm is proportional to $q^2 + \xi^{-2}$. Then by the same reasoning as for the Gaussian model,

$$f_{\text{sing.}} \propto \xi^{-d} \propto (t_c - t)^{d/2}.$$

  All further critical exponents come out exactly the same.
- Like the Gaussian model, the phantom loop approximation is unstable below the critical point. For $t > t_c$, loops of greater length are favored more, and it becomes favorable for loops to retrace their steps over and over. In the true Ising model, this instability is stabilized by the fact that loops can't self-intersect.
- The Gaussian model and the phantom loop approximation contain essentially the same physics; their difference is analogous to the difference between wavefunction evolution and the path integral.

- The phantom loop approximation can also explain why there is no phase transition in 1D. We can think of each paths as carrying information with fidelity t^ℓ. In this case of the 1D Ising model, this means all correlation functions decay exponentially.
- We can also rederive the critical exponent ν heuristically. Paths shorter than the characteristic path length $\bar{\ell} = -1/\log(2dt) \sim (t_c - t)^{-1}$ have good information transfer, and since the paths are random walks, their diameter goes as $\bar{\ell}^{1/2}$. Then the sizes of correlated regions diverge as $(t_c - t)^{-1/2}$, exactly the divergence of the correlation length we saw before.
- The Gaussian model gives incorrect critical exponents for $d < 4$. To see why, note that random walks have Hausdorff dimension 2. Then two random walks will generically intersect if $d < 2+2$. As a result, we must account for intersections, which correspond to accounting for the um^4 term in the Gaussian model. For example, forbidding self-interactions swells the size of random walks, so $\nu > 1/2$ for $d < 4$.

6.7 Monte Carlo

In this section, we take a detour to describe how lattice models are simulated.

- At heart, lattice models like the Ising model don't have time dependence. The Hamiltonian is not the generator of time translation as in Hamiltonian mechanics; all it does is assign energies to configurations. (One can change this by specifying a Poisson bracket/commutator structure, e.g. the standard angular momentum relations for the $O(n)$ model, but this is an additional layer of structure.)
- In thermal equilibrium, these configuration energies correspond to probabilities, but the model says nothing about how equilibrium is physically approached, or how the system evolves between states over time. However, most of the quantities we are concerned about, such as critical exponents, are independent of such details.
- Thus, when simulating the system, we are free to substitute any dynamics we want, no matter how realistic, as long as the resulting probability distribution is correct. We can think of a simulation as simply a way to sample from the canonical ensemble.
- One simple example of such an algorithm for the Ising model is as follows.

  1. At each step, pick a random spin i.
  2. Compute the energies $E_\pm$ for $m_i = \pm 1$, holding all other spins fixed.
  3. Assign i to be spin up or down, with probability $p_\pm = e^{-\beta E_\pm}/(e^{-\beta E_+} + e^{-\beta E_-})$.

  Below, we will develop a theory that explains why this algorithm works correctly.

All of the algorithms we will consider will be Markov chains.

- A Markov chain is a stochastic process which, at each time step, randomly evolves the state of the system in a way that only depends on the current state. Thus, a Markov chain can be represented by a matrix whose elements $P_{\beta\alpha}$ represent the probability of a transition $\alpha \to \beta$.

- If the probabilities of occupancy of the states are ρ_α, then after a timestep,

$$\rho'_\beta = \sum_\alpha P_{\beta\alpha}\rho_\alpha.$$

  For the new probabilities to make sense, we require

$$0 \leq P_{\beta\alpha} \leq 1, \quad \sum_\beta P_{\beta\alpha} = 1.$$

  These enforce positivity and normalization of probability, respectively.

- Note that P is generally not symmetric, so we cannot just apply, e.g. the spectral theorem. For each distinct eigenvalue λ (defined as the diagonal elements in Jordan normal form), there is at least one corresponding right eigenvector and left eigenvector. However, these eigenvectors generally will not coincide, nor will there be a complete basis of eigenvectors.

- Note that normalization of probability tells us there is a left eigenvector with eigenvalue 1, namely $(1, 1, \ldots, 1)$. Thus, there exists at least one right eigenvector with eigenvalue 1.

- In addition, any right eigenvector $\boldsymbol{\rho}$ with eigenvalue $\lambda \neq 1$ must have components that sum to zero. To see this, note that

$$\lambda \sum_\beta \rho_\beta = \sum_{\alpha,\beta} P_{\beta\alpha}\rho_\alpha = \sum_\alpha \rho_\alpha$$

  where we again used normalization of probability.

- We would like to ensure that our simulation reaches a unique stationary probability distribution, but this requires imposing more conditions. For example, the Markov chain might have totally disconnected components. A more subtle issue is that it may have deterministic cycles, like $\alpha \to \beta \to \gamma \to \alpha$, through which probability continually shuffles.

- To prevent these issues, we restrict to ergodic Markov chains, defined as Markov chains which do not have cycles, and which are irreducible, meaning that one can get from any state to any other in a finite sequence of moves. This definition is suited for what we will need below, but other definitions are used in physics and mathematics.

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Note. Monte Carlo methods are used in many other fields. A more general sense of "Monte Carlo" is simply that of sampling from a given probability distribution. These distributions in general don't have the nice structure above, so other ideas are necessary.

It is easy and fast to sample from a uniform distribution, using a standard pseudorandom number generator. This leads to a trivial way to sample from an arbitrary one-dimensional distribution if its inverse cdf is known: we simply sample the percentile rank (in $[0, 1]$) and apply the inverse cdf function. However, evaluating the cdf is often intractable analytically – and if we could do it easily numerically, we probably wouldn't need to be doing Monte Carlo in the first place. Furthermore, it's not clear how this extends to higher-dimensional distributions.

A better, but still simple idea is von Neumann's accept-reject method. Suppose that the distribution is bounded between $x_{\min}$ and $x_{\max}$, and the probability density has a maximal value

$f_{\max}$. Then we draw a number x uniformly in $[x_{\min}, x_{\max}]$, then accept it with probability $f(x)/f_{\max}$. Intuitively, this works because we can think of drawing the graph of the distribution on a piece of paper, and throwing darts at that piece of paper. It also clearly works for higher-dimensional distributions. However, it can be inefficient if there are too many rejections, which occurs for sharply peaked functions. The method can be refined by drawing from any distribution which bounds the desired distribution more closely.

Note. In general, Monte Carlo methods are better when dealing with high-dimensional distributions. If N describes the total computation time, the error of Monte Carlo methods falls as $1/\sqrt{N}$. A one-dimensional integral can be computed with the trapezoidal rule with error improving as $1/N^2$, but a d-dimensional integral has error improving as $1/N^{2/d}$, which is less efficient than Monte Carlo for high d. For lattice models we are always working with d large, in fact thermodynamically large. Monte Carlo is also very useful in particle physics, where it is used to simulate particle collisions and their subsequent detection; these are physical processes involving a large number of random choices.

7 Continuous Spins

7.1 The Nonlinear σ Model

Next, we consider low-temperature expansions for continuous spins. Since the low-energy excitations are Goldstone modes, rather than droplets, the most useful low-temperature description is a field theory of these Goldstone modes. We already know that these modes destroy long-range order in $d \leq 2$. Then for d just above 2, the critical temperature is low, so a low-temperature expansion can probe critical phenomena.

- We consider n-component unit spins $\mathbf{s}(i)$ on a lattice, with

$$-\beta H = K \sum_{\langle ij \rangle} \mathbf{s}(i) \cdot \mathbf{s}(j) = K \sum_{\langle ij \rangle} \left(1 - \frac{(\mathbf{s}(i) - \mathbf{s}(j))^2}{2} \right).$$

- At low temperatures, the fluctuations between nearby spins are small, so we may replace the difference with a gradient,

$$-\beta H = -\frac{K}{2} \int d^d\mathbf{x} \, (\nabla \mathbf{s}(\mathbf{x}))^2 + \text{const.}$$

  As usual, this implicitly imposes a cutoff $\Lambda \sim \pi/a$ on the field $\mathbf{s}(\mathbf{x})$.

- The integration measure $\mathcal{D}\mathbf{s}(\mathbf{x})$ for the partition function is constrained to spins with unit norm. To remove this constraint, suppose the ground state is $\mathbf{s}(\mathbf{x}) = (0, \ldots, 1)$ and let

$$\mathbf{s}(\mathbf{x}) = (\boldsymbol{\pi}(\mathbf{x}), \sigma(\mathbf{x}))$$

  where $\boldsymbol{\pi}$ is an $n-1$ component vector representing the transverse Goldstone modes. Then

$$\int d\boldsymbol{\pi} d\sigma \, \delta(\pi^2 + \sigma^2 - 1) = \int d\boldsymbol{\pi} d\sigma \, \delta \left[(\sigma - \sqrt{1-\pi^2})(\sigma + \sqrt{1-\pi^2}) \right] = \int \frac{d\boldsymbol{\pi}}{2\sqrt{1-\pi^2}}.$$

  We have thrown out the solution with negative σ, since we're only considering low-energy excitations. For the same reason, we can formally extend the integration over $\boldsymbol{\pi}$ to $(-\infty, \infty)$.

- Dropping the factor of 2, we see that every lattice site contributes a factor of $1/\sqrt{1-\pi^2}$ to the measure. Thus the Hamiltonian density is modified to

$$\beta \mathcal{H} = \frac{K}{2} (\nabla \boldsymbol{\pi})^2 + \frac{K}{2} (\nabla \sqrt{1-\pi^2})^2 + \frac{\rho}{2} \log(1-\pi^2)$$

  where $\rho = N/V = 1/a^d$ is the density of lattice points.

- We see that by integrating out the σ, we get complicated nonlinear terms for the pions, causing interactions between Goldstone modes; hence the term 'nonlinear σ model'.

- To handle the computations, we expand to lowest nontrivial order. We expect $\langle \pi^2 \rangle \sim 1/K$ from the zeroth order term, so the next terms are $O(K\pi^4)$ or $O(\pi^2)$, giving

$$\beta \mathcal{H}_0 = \frac{K}{2} (\nabla \boldsymbol{\pi})^2, \quad \mathcal{U}_1 = \frac{K}{2} (\boldsymbol{\pi} \cdot \nabla \boldsymbol{\pi})^2 - \frac{\rho}{2} \pi^2.$$

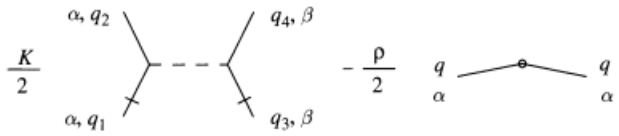
  This is a low-temperature expansion because $K \sim 1/T$, so we are expanding in powers of T.

Next, we perform a first-order renormalization group analysis.

- In Fourier space, we have
$$\beta H_0 = \frac{K}{2} \int đ\mathbf{q}\, q^2 |\boldsymbol{\pi}(\mathbf{q})|^2$$
and
$$U_1 = -\frac{K_{\text{int}}}{2} \int đ\mathbf{q}_1\, đ\mathbf{q}_2\, đ\mathbf{q}_3\, \pi_\alpha(\mathbf{q}_1)\pi_\alpha(\mathbf{q}_2)\pi_\beta(\mathbf{q}_3)\pi_\beta(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)(\mathbf{q}_1\cdot\mathbf{q}_3) - \frac{\rho}{2}\int đ\mathbf{q}\, |\boldsymbol{\pi}(\mathbf{q})|^2$$
where we renamed the K in U_1 to K_{int} for later convenience.

- As usual, we split the modes into $\boldsymbol{\pi}^<$ and $\boldsymbol{\pi}^>$, corresponding to momenta above and below Λ/b. As shown earlier, the renormalized Hamiltonian picks up a term $\langle U_1[\boldsymbol{\pi}]\rangle_0^>$ where the expectation value is taken with respect to βH_0 and integrates over the $\boldsymbol{\pi}^>$ modes.

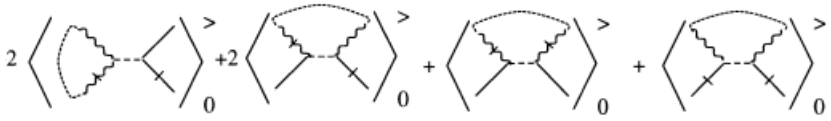
- The two perturbation terms can be represented as follows.



Here, the notches represent momentum factors, due to the $\mathbf{q}_1\cdot\mathbf{q}_3$ factor above. We perform casework on each external leg, showing it as a wavy line if its momentum is above Λ/b.

- The ρ interaction contributes three types of diagrams, with zero, one, and two $\boldsymbol{\pi}^>$ external legs. The first simply reproduces the ρ term for the $\boldsymbol{\pi}^<$ fields, the second is zero by symmetry, and the third contributes additively to the free energy, so we ignore it.

- The K_{int} interaction is more interesting, with 16 diagrams. Again, nontrivial effects can only come from graphs with two $\boldsymbol{\pi}^>$ and two $\boldsymbol{\pi}^<$ external legs, shown below.



The first two diagrams are zero by symmetry, as we have one $\boldsymbol{\pi}^>$ external leg with a momentum factor in each case. The other two renormalize ρ and K respectively.

- Performing the calculation, we find
$$\widetilde{K} = K\left(1 + \frac{I_d(b)}{K}\right), \quad I_d(b) = \int_{\Lambda/b}^{\Lambda} \frac{đ\mathbf{k}}{k^2} = \frac{K_d \Lambda^{d-2}(1-b^{2-d})}{d-2}$$
where we used $K = K_{\text{int}}$. The renormalization of ρ isn't as interesting, as it merely tracks the density of points. We may also ignore the renormalization of K_{int}, because it must always be equal to K by rotational symmetry.

- Finally, rescaling and renormalizing, we have
$$K' = b^{d-2}\zeta^2 \widetilde{K}$$
where we have divided the spin values by ζ, which is specified by demanding the renormalized spins also have unit norm.

- After some simplification, we find that the temperature $T = K^{-1}$ scales as

$$\frac{dT}{d\ell} = -(d-2)T + (n-2)K_d\Lambda^{d-2}T^2$$

where $b = 1 + \ell$. For $d > 2$, the low-temperature fixed point is stable, yielding an ordered phase. For $d < 2$ there is no ordered phase, as expected by the Mermin–Wagner theorem.

- For $d = 2$, the situation is more subtle. For $n > 2$ there is no ordered phase and for $n < 2$ there is an ordered phase (e.g. 2D Ising). However, for $n = 2$, we have $dT/d\ell = 0$, which is quite special; one can also show this remains true to all orders.

7.2 ∗ The Nonlinear σ Model, Redux

Before continuing, we recap the physics of SSB in the $O(n)$ model in dimension d.

- If we ignore thermal fluctuations entirely, then for any d and n we find spontaneous symmetry breaking below the critical temperature, and for $n > 1$ this means there are Goldstone modes, which we parametrize by θ.

- The Goldstone modes only appear in the free energy through derivative terms, such as $(\nabla\theta)^2$, implying the existence of long-range correlations everywhere below the critical point. Moreover, Goldstone modes are soft/gapless, i.e. their free energy vanishes in the limit $k \to 0$ with fixed amplitude. In relativistic quantum field theory, that means they are massless.

- Now we consider the effect of thermal fluctuations. For $d = n = 1$ and $d = 2, n \geq 2$, thermal fluctuations set $\langle\phi\rangle = 0$. More precisely, the correlator $\langle(\theta(\mathbf{x}) - \theta(0))^2\rangle$ grows without bound for high $\mathbf{x}$, as we computed earlier. This means that if we set ϕ to some value ϕ_0 as a boundary condition, its value far from that boundary will be essentially independent of ϕ_0.

- The absence of gapless modes for $d = 1$ can be understood using the analogy with quantum mechanics. The nonlinear sigma model translates to a quantum particle on a sphere, whose energy levels are gapped. Since the spectrum is the same in both cases, there are no gapless modes in the original statistical field theory.

- For $d > 2$, where Goldstone modes exist, the spherical symmetry strongly constrains the coefficients in the free energy. In particular, for $T < T_c$ the Goldstone modes automatically have $\eta = 0$, though η is generally nonzero at $T = T_c$.

- We know that whenever SSB occurs, we have gapless modes. In general, we've seen that there are *not* gapless modes whenever SSB doesn't occur. The surprising part is that for $d = n = 2$, we still have a phase transition and gapless modes!

To understand this better, it's useful to focus on only the Goldstone modes.

- Regardless of the presence of SSB, at low temperatures, $\langle|\phi|\rangle$ is approximately equal to its mean-field value, because longitudinal excitations are gapped, so well below T_c it must be nonzero. In particular, for very low temperatures we should be able to set $\langle|\phi|\rangle$ exactly to its mean field value and only consider the Goldstone modes. Here we've just using 'Goldstone mode' to refer to the transverse modes, which may or may not be gapped.

- The resulting theory is a nonlinear sigma model, i.e. a theory where the fields are coordinates on a manifold. The ϕ^2 term and ϕ^4 interaction just become constants and don't matter. Instead, the nontrivial physics comes entirely from the kinetic term!
- For $n = 2$, which we call the XY model, there is just a single Goldstone mode. But for $n = 3$ the field is parametrized by spherical coordinates (θ, ϕ), giving a lowest-order gradient term

$$(\nabla \boldsymbol{\phi})^2 \to (\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2$$

  and the second term is an interaction between θ and ϕ. In general, the kinetic term for a nonlinear sigma model can be read off from the metric. In this case, the fields live on a sphere, and the strength of the interactions is determined by the curvature of the sphere.
- It would seem that the Goldstone modes are clearly gapless, no matter what the values of d and n are, but this is too naive: the gaplessness of a mode can only be read off the free energy if the coupling is weak. We will find that under RG flow, the coupling gets stronger and stronger in $d \leq 2$. The exception will be $d = n = 2$, where the coupling stays the same.

With this background in mind, we reintroduce the nonlinear sigma model.

- We work at $T < T_c$, ignore the longitudinal mode and rescale the field to have unit norm,

$$F[\mathbf{n}] = \int d\mathbf{x}\, \frac{1}{2e^2} \nabla\mathbf{n} \cdot \nabla\mathbf{n}, \quad e^2 = \langle|\phi|\rangle^{-2}.$$

  Here we've redefined $K = 1/e^2$, and the partition function is

$$Z = \int \mathcal{D}\mathbf{n}\, \delta(\mathbf{n}(\mathbf{x})^2 - 1) e^{-F[\mathbf{n}]}.$$

  While the theory looks like a free theory, interactions occur due to the constraint. Note that the field is not canonically normalized; the theory is described by the single parameter e, and it's useful for us to put it in the coefficient of the kinetic term.
- For $e^2 \ll 1$, the original ϕ fields live on a large sphere, so the curvature is small and the Goldstone modes interact weakly. For $e^2 \gg 1$, the Goldstone modes are instead strongly coupled and $|\phi|$ is small. Since we are ignoring the longitudinal mode, our model is only a good description of the physics for e^2 small, so we will expand perturbatively in e^2. Another piece of intuition is that F is really βF, so $e^2 \sim T$. Then the weak coupling limit is also the low temperature limit.
- In the previous section, we separated out a single direction,

$$\mathbf{n}(\mathbf{x}) = (\boldsymbol{\pi}(\mathbf{x}), \sigma(\mathbf{x}))$$

  and then integrated out the σ and expanded in a Taylor series. However, this completely destroys the underlying spherical symmetry.
- Instead, we use the background field method. A general field configuration can be written as

$$n^a(\mathbf{x}) = \tilde{n}^a(\mathbf{x})(1 - \chi(\mathbf{x})^2)^{1/2} + \sum_{\alpha=1}^{N-1} \chi_\alpha(\mathbf{x}) e_\alpha^a(\mathbf{x})$$

where $\tilde{n}^a(\mathbf{x})$ is a 'background field' and the $e^a_\alpha(\mathbf{x})$ are frame fields satisfying

$$\tilde{n} \cdot \tilde{n} = 1, \quad \tilde{n} \cdot e_\alpha = 0, \quad e_\alpha \cdot e_\beta = \delta_{\alpha\beta}.$$

Both the background field and frame field vary slowly in space. The functions $\chi_\alpha(\mathbf{x})$ parametrize the short-wavelength fluctuations, like our usually 'fast/slow' decomposition but in real space.

- Next, we expand the free energy perturbatively in χ. This is because $\chi^2 \sim e^2$, as can be checked by scaling back to the ϕ fields or by evaluating the integral by saddle. First,

$$\begin{aligned}\nabla n^a &= \nabla \tilde{n}^a \sqrt{1-\chi^2} + \tilde{n}^a \nabla \sqrt{1-\chi^2} + \nabla(\chi_\alpha e^a_\alpha) \\ &= \nabla \tilde{n}^a (1 - \chi^2/2) - \tilde{n}^a \chi_\alpha \nabla \chi_\alpha + \nabla(\chi_\alpha e^a_\alpha) + O(\chi^2).\end{aligned}$$

where the ∇ carries an implicit vector index.

- Therefore, the gradient term is

$$(\nabla n^a)^2 = (\nabla \tilde{n}^a)^2(1-\chi^2) + (\nabla \chi_\alpha)^2 + \chi_\alpha \chi_\beta \nabla e^a_\alpha \nabla e^a_\beta + 2\nabla \tilde{n}^a \nabla(\chi_\alpha e^a_\alpha) + O(\chi^3).$$

Here the vector indices between the gradients are implicitly contracted, and we used $\tilde{n}^a \nabla \tilde{n}^a = 0$ to remove a cross-term.

With this setup, we perform the RG transformation by integrating out the χ.

- The partition function is

$$Z = \int \mathcal{D}\tilde{n}\, \delta(\tilde{n}^2 - 1) e^{-F_0[\tilde{n}]} \langle e^{-F_I[\tilde{n},\chi]} \rangle_0, \quad F_0[\tilde{n}] = \frac{1}{2e^2} \int d\mathbf{x}\, (\nabla \tilde{n})^2$$

where the expectation value is taken over the χ in the Gaussian ensemble, and

$$F_I[\tilde{n}, \chi] = \frac{1}{2e^2} \int d\mathbf{x}\, \left(-\chi^2 (\nabla \tilde{n}^a)^2 + \chi_\alpha \chi_\beta \nabla e^a_\alpha \nabla e^a_\beta + 2\nabla \tilde{n}^a \nabla(\chi_\alpha e^a_\alpha) \right).$$

- As usual, we expand the exponential (where the expansion variable is χ), for

$$\langle e^{-F_I} \rangle_0 = 1 - \langle F_I \rangle_0 + \frac{1}{2} \langle F_I^2 \rangle_0 + \dots.$$

We are only interested in the renormalization of the coupling e, so the first term suffices. Higher order terms generate more and more additional terms in the free energy, but they all turn out to be irrelevant. One might think that the $\langle F_I^2 \rangle_0$ term renormalizes the gradient term at $O(\chi^2)$ due to the linear term in F_I, but this turns out to not occur.

- At lowest order, applying Wick's theorem, the linear term does not contribute and

$$\langle F_I \rangle_0 = \frac{1}{2e^2} \int d\mathbf{x}\, (-\delta_{\alpha\beta} (\nabla \tilde{n}^a)^2 + \nabla e^a_\alpha \nabla e^a_\beta) \langle \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{x}) \rangle_0.$$

We now choose χ to contain the Fourier modes above Λ/ζ, so that

$$\langle \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{x}) \rangle_0 = e^2 \delta_{\alpha\beta} I_d, \quad I_d = \int_{\Lambda/\zeta}^{\Lambda} \frac{đ\mathbf{q}}{q^2} = \frac{\Omega_{d-1}}{(2\pi)^d} \Lambda^{d-2} \times \begin{cases} \zeta - 1 & d = 1 \\ \log \zeta & d = 2 \\ 1 - \zeta^{2-d} & d \geq 3 \end{cases}$$

by our earlier work. Here the factor of e^2 is because the kinetic term for χ comes with a factor of $1/e^2$. Then all powers of χ come with powers of e, justifying the expansion in χ.

- The term $\nabla e^a_\alpha \nabla e^a_\beta$ also contributes to the gradient. To see this, use the completeness relation

$$\tilde{n}^a \tilde{n}^b + e^a_\alpha e^b_\alpha = \delta^{ab}$$

and $\tilde{n}^a e^a_\alpha = 0$, which implies $\tilde{n}^a \nabla e^a_\alpha = -(\nabla \tilde{n}^a) e^a_\alpha$, for

$$\begin{aligned}\nabla e^a_\alpha \nabla e^a_\alpha &= \nabla e^a_\alpha \nabla e^b_\alpha (\tilde{n}^a \tilde{n}^b + e^a_\beta e^b_\beta) \\ &= e^a_\alpha e^b_\alpha (\nabla \tilde{n}^a)(\nabla \tilde{n}^b) + (e^a_\beta \nabla e^a_\alpha)(e^b_\beta \nabla e^b_\alpha) \\ &= \nabla \tilde{n}^a \nabla \tilde{n}^a + (e^a_\beta \nabla e^a_\alpha)(e^b_\beta \nabla e^b_\alpha)\end{aligned}$$

where the first term is the gradient, and the second term is a new, irrelevant 'torsion' term.

- Finally, applying $\delta_{\alpha\beta}\delta^{\alpha\beta} = N - 1$, we conclude

$$\langle F_I \rangle_0 = (2 - N) I_d \int d\mathbf{x}\, \frac{1}{2} (\nabla \tilde{n}^a)^2.$$

At the linear order we're working, $\langle e^{-F_I} \rangle_0 = e^{-\langle F_I \rangle_0}$, so we have

$$\frac{1}{e'^2} = \frac{1}{e_0^2} + (2 - N) I_d.$$

- We don't have to rescale the fields, since we aren't maintaining canonical normalization; we only need to scale the lengths by $1/\zeta$. Since $[e^2] = 2 - d$ we conclude

$$\frac{1}{e^2(\zeta)} = \zeta^{d-2} \left(\frac{1}{e_0^2} + (2 - N) I_d \right).$$

Note there is no running for $N = 2$, because in this case there is a single free Goldstone mode.

Next, we step back and look at our result.

- For $d < 2$, the coupling is relevant. That is, it gets stronger in the IR, so in the IR limit our perturbative expansion breaks down and we cannot determine whether the theory has gapless modes. By our earlier argument, we know there aren't.

- Defining $\xi = e^s$, the beta function for $d = 2$ is

$$\beta(e) = \frac{de}{ds} = (N - 2) \frac{e^3}{4\pi}.$$

In this case, the coupling e^2 is marginally relevant, in contrast to the marginally irrelevant coupling u at the Gaussian fixed point for $d = 4$. Then the coupling is weak in the UV limit, so we say the theory is asymptotically free. In general, an asymptotically free theory is one which has no irrelevant operators at all, only relevant and marginally relevant ones. They are special and rare; Yang-Mills is an important example.

- In the case $d = N = 2$, the coupling does not run, and one can show that it does not run to all orders. Then there do exist gapless modes, and hence a phase transition does occur, but it is not a symmetry breaking transition.

- Finally, we can expand in $d = 2 + \epsilon$, where

$$\frac{de}{ds} = -\frac{\epsilon e}{2} + (N-2)\frac{e^3}{4\pi}\Lambda^\epsilon.$$

This has a perturbative fixed point,

$$e_\star^2 = \frac{2\pi\epsilon}{N-2}\Lambda^{-\epsilon}.$$

However, the fixed point is unstable, so it is a UV fixed point rather than an IR fixed point.

- To understand this, note that if we start from the Wilson–Fisher fixed point and 'lower the temperature', we flow away from it to the zero-temperature fixed point, i.e. the ordered phase. In our sigma model, we've started in this ordered phase; by running our RG flow in reverse we can get back to the Wilson–Fisher fixed point!

- We can use the $d = 2 + \epsilon$ expansion to find the critical exponents at the Wilson–Fisher fixed point. Identifying $e^2 = t$, we have

$$\frac{dt}{ds} = -\epsilon t + (N-2)\frac{t^2}{2\pi}\Lambda^\epsilon$$

and linearizing about the fixed point gives $\Delta_t = \epsilon$, which implies $\nu = 1/\epsilon$. This isn't very accurate in $d = 3$, but we can match this with the $d = 4 - \epsilon$ expansion to get better results.

We can show that gapless modes do not exist for $d = 2$ and large N with the large N expansion.

- We rewrite the partition function in the unconstrained form

$$Z = \int \mathcal{D}n\mathcal{D}\sigma \, \exp\left(-\frac{1}{2e_0^2}\int d\mathbf{x}\, \nabla\mathbf{n}\cdot\nabla\mathbf{n} - \frac{i}{2e_0^2}\int d\mathbf{x}\, \sigma(\mathbf{n}\cdot\mathbf{n} - 1)\right).$$

The σ field is a Lagrange multiplier; integrating it out recovers the unit length constraint.

- However, since the free energy is quadratic in $\mathbf{n}$, we can instead integrate out $\mathbf{n}$, giving

$$Z = \int \mathcal{D}\sigma \, \det{}^{N/2}(-\nabla^2 + i\sigma(\mathbf{x})) \exp\left(\frac{i}{2e_0^2}\int d\mathbf{x}\, \sigma\right).$$

Using $\det A = \exp \operatorname{tr} \log A$, we can write this as part of the free energy density,

$$Z = \int \mathcal{D}\sigma \, \exp\left(-\frac{N}{2}\operatorname{tr}\log(-\nabla^2 + i\sigma) + \frac{i}{2e_0^2}\int d\mathbf{x}\, \sigma\right).$$

Note that there is an implicit UV cutoff on the functional traces and logs.

- Since N is large, we can apply saddle point integration. However, we really need both terms to be large, so we send $N \to \infty$ and $e_0^2 \to 0$, keeping $e_0^2 N$ fixed. Applying the identity

$$\delta \operatorname{tr}\log X = \operatorname{tr} X^{-1}\delta X$$

the saddle point for σ satisfies

$$G(\mathbf{x},\mathbf{x}) = \frac{1}{e_0^2 N}$$

where $G(\mathbf{x},\mathbf{x}')$ is the Green's function for $(-\nabla^2 + i\sigma(\mathbf{x}))$.

- We guess a constant solution, of the form $\sigma(\mathbf{x}) = -i\mu^2$. Then in Fourier space,

$$\int^{\Lambda} \frac{đ\mathbf{k}}{k^2+\mu^2} = \frac{1}{e_0^2 N}.$$

  Integrating to solve for μ, in $d = 2$ we have

$$\frac{1}{4\pi}\log\left(\frac{\Lambda^2+\mu^2}{\mu^2}\right) = \frac{1}{e_0^2 N}.$$

  Assuming we started with a weakly coupled UV theory, $e_0^2 N \ll 1$, then we can self-consistently assume $\mu \ll \Lambda$ to find

$$\mu \approx \Lambda e^{-2\pi/e_0^2 N}.$$

- Now, the saddle point equations for σ tell us which configurations of σ count, when we integrate it out in the full path integral. Since a constant is a solution, integrating out the σ gives a quadratic term for $\mathbf{n}$, so the Goldstone bosons are no longer gapless for large N.

- One can show there are no gapless modes for all $N \geq 3$ and hence no phase transition. The effect is nonperturbative, since the Taylor expansion of μ in e_0^2 is identically zero.

- In the context of quantum field theory, the appearance of the dimensionful scale μ from a theory with only dimensionless couplings e_0^2 is called dimensional transmutation. In the Wilsonian picture, it is obvious that this scale really comes from the UV cutoff Λ.

- In the case $d > 2$, the same manipulations give

$$\frac{1}{e_0^2 N} = \int^{\Lambda} \frac{đ\mathbf{k}}{k^2+\mu^2} \sim \begin{cases} \Lambda^{d-2} - \mu^2\Lambda^{d-4} & d \geq 4, \\ \Lambda^{d-2} - \mu^{d-2} & 2 < d < 4 \end{cases}$$

  where we took the two leading terms in the integral, i.e. by expanding in a series in μ/Λ. Then for sufficiently weak coupling, $e_0^2 N \lesssim \Lambda^{2-d}$, there is no solution for μ, indicating that a phase transition does occur, giving gapless Goldstone modes.

- Finally, we can extract another critical exponent from this result. We have $T - T_c \sim e^2 - e_\star^2$, where $e_\star$ is the critical coupling where $e_\star^2 N \sim \Lambda^{2-d}$, and $\xi \sim 1/\mu$. Then

$$T - T_c \sim \begin{cases} \xi^{-2} & d \geq 4, \\ \xi^{2-d} & 2 < d < 4, \end{cases} \qquad \nu = \begin{cases} 1/2 & d \geq 4, \\ 1/(d-2) & 2 < d < 4 \end{cases}$$

  where the former result agrees with mean field theory, and the latter agrees with the $d = 2 + \epsilon$ expansion and with the $d = 4 - \epsilon$ expansion at large N.

Note. In a quantum field theory, gapless modes must appear for a phase transition to occur. This is because phase transitions separate distinct vacua, but if there were a gap, it would be possible to adiabatically transition from one to the other. The same statement holds in the statistical field theory, as gapless modes are gapless on both sides of the correspondence.

7.3 The XY Model

In the previous section, we have shown that the $n = d = 2$ continuous spin model, called the XY model, has a strange behavior under RG. Numerical evidence suggests that there is a divergence of the susceptibility at finite temperature, indicating a phase transition. However, as Mermin–Wagner still applies, there can be no local order parameter.

- As for the lattice gauge theory, we are motivated to look at correlation functions, in particular the decay of $\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle$. We have

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle = \frac{1}{Z} \prod_{i=1}^{N} \left(\int_0^{2\pi} \frac{d\theta_i}{2\pi} \right) \cos(\theta_0 - \theta_r) e^{K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

  and at high temperatures we can expand to lowest order in K,

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle = \frac{1}{Z} \prod_{i=1}^{N} \left(\int_0^{2\pi} \frac{d\theta_i}{2\pi} \right) \cos(\theta_0 - \theta_r) \prod_{\langle ij \rangle} \left[1 + K \cos(\theta_i - \theta_j) + O(K^2) \right].$$

  Again, the diagrams that contribute are closed graphs with endpoints at 0 and r, as all other diagrams contain a factor of $\int_0^{2\pi} \cos(\theta)\, d\theta = 0$. Since we are expanding to lowest order in K, only the shortest paths from 0 to r matter.

- Note that every internal point gives a factor of $1/2$, as

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \frac{1}{2} \cos(\theta_1 - \theta_3).$$

  Therefore we have

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle \approx \left(\frac{K}{2} \right)^r = e^{-r/\xi}, \quad \xi = \frac{1}{\log(2/K)}.$$

  In general there may be multiple shortest length paths, but there are at most exponentially many, so the decay of the correlation function is still exponential.

- At low temperatures, the Hamiltonian is approximately $\mathcal{H} = K(\nabla\theta)^2/2$ and the fluctuations are Gaussian. Then we can use the same reasoning as earlier; using the identity

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle = \langle \mathrm{Re}\, e^{i(\theta_0 - \theta_r)} \rangle = \mathrm{Re} \exp \left[-\frac{1}{2} \langle (\theta_0 - \theta_r)^2 \rangle \right]$$

  and the fact that fluctuations in two dimensions grow as

$$\frac{1}{2} \langle (\theta_0 - \theta_r)^2 \rangle = \frac{1}{2\pi K} \log(r/a)$$

  we conclude that

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle \approx \left(\frac{a}{r} \right)^{1/2\pi K}.$$

  Then at low temperatures, we have 'quasi long-range order', as correlations decay slowly.

- Since θ is dimensionless, we have an anomalous dimension

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_r \rangle \propto \frac{1}{r^\eta}, \quad \eta = \frac{1}{2\pi K} = \frac{e^2}{2\pi}.$$

- Since the decay of correlations is exponential at higher temperatures and algebraic at low temperatures, we expect there is a phase transition in between. However, this is a cheat, because the low temperature expansion never used the fact that $n = 2$! The decay of correlations is always algebraic at zero temperature, but for $n > 2$, the zero-temperature fixed point is unstable, so we don't get algebraic decay at finite temperature.
- The case $n = 2$ is more interesting. Since $dT/d\ell = 0$ at all orders, we have an RG fixed point for every temperature. At every one of these points, the decay of correlations is algebraic.
- Kosterlitz and Thouless showed that these fixed points switch from stable to unstable at a critical temperature T_c, corresponding to the XY model's expected phase transition. Above T_c, we RG flow to the high-temperature fixed point, which has exponential decay.

As we'll show below, the instability is due to the proliferation of topological defects.

- The XY model supports topological defects ($\pi_1(S^1) = \mathbb{Z}$) indexed by the topological charge

$$n = \frac{1}{2\pi} \int (\nabla\theta) \cdot d\mathbf{s}.$$

  Note that this expression only makes sense for a coarse-grained field. Our gradient expansion for the nonlinear σ model can't detect these defects, because they only look at small deformations about the ground state.

- For a symmetric vortex of charge n, obeying the equation of motion $\nabla^2\theta = 0$, we have

$$\nabla\theta = n(-y/r^2, x/r^2, 0) = -\frac{n}{r}(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) = -n\nabla \times (\hat{\mathbf{z}} \log r).$$

  Note that while the $\nabla\theta$ configurations of vortices are intuitive, simply pointing along $\pm\hat{\boldsymbol{\theta}}$ for charge ± 1, the θ configurations are not. The θ configurations of the $+1$ vortices point radially or clockwise or counterclockwise, while the -1 vortices point along hyperbolas.

- In the context of superfluids, $\mathbf{v} = \nabla\theta$ is the velocity of the superfluid, so the vortices are literally fluid vortices with clockwise or counterclockwise flow. The equation of motion is $\nabla \cdot \mathbf{v} = 0$, so the flow is incompressible.

- The approximation above breaks down at some length scale a, which we call the vortex core size, so the total energy is

$$\beta\mathcal{E}_n = \beta\mathcal{E}_n^0 + \frac{K}{2} \int_a^\infty d\mathbf{x}\, (\nabla\theta)^2 = -\beta\mathcal{E}_n^0 - \pi K n^2 \log(L/a)$$

  where $\mathcal{E}_n^0$ is the energy contribution from the core region.

- The total energy diverges with system size, but this doesn't mean vortex formation is impossible, because the number of vortex configurations also diverges with system size. In particular, the partition function for one vortex is

$$Z_1 \sim \left(\frac{L}{a}\right)^2 e^{-\beta\mathcal{E}_n} \sim \left(\frac{L}{a}\right)^{2-\pi K n^2}.$$

  Therefore, for $K < 2/\pi$, the spontaneous formation of vortices with unit charge is favored; past this point, our expression Z_1 isn't useful as there will generally be many vortices.

- If vortices didn't interact, this would be the end of the analysis. However, vortex configurations can substantially lower their energy by interaction; for example, a dipole of charges ± 1 with separation $\mathbf{d}$ has gradient

$$\nabla\theta \approx \mathbf{d} \cdot \nabla \left(\frac{\hat{\mathbf{r}} \times \hat{\mathbf{z}}}{r} \right)$$

  at large distances, giving a finite energy. Therefore, dipoles generally appear at any finite temperature. The XY phase transition is due to dipoles unbinding into a 'plasma' of vortices.

- As in lattice gauge theory, this transition can be understood in terms of nonlocal quantities. Let $Q(\ell)$ be the topological charge in a square of side ℓ. Then in the low-temperature phase,

$$\langle Q(\ell)^2 \rangle \propto \ell$$

  due to dipoles straddling the perimeter, while in the high-temperature phase

$$\langle Q(\ell)^2 \rangle \propto \ell^2$$

  since all the charges in the loop are independent.

To make further progress, we have to understand the interaction of vortices.

- Naively, since $\mathbf{v} = \nabla\theta$ we must have $\nabla \times \mathbf{v} = 0$, so closed loop integrals of $\mathbf{v}$ must always be zero; the fact that vortices exist implies that there must be singularities in θ, and hence singularities in $\mathbf{v}$. Concretely, for vortices of charge n_i at locations $\mathbf{x}_i$, we require

$$\nabla \times \mathbf{v} = 2\pi\hat{\mathbf{z}} \sum_i n_i \delta^2(\mathbf{x} - \mathbf{x}_i).$$

  Therefore we can separate $\mathbf{v}$ into a singular and nonsingular part,

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \quad \mathbf{v}_0 = \nabla\phi$$

  where ϕ is nonsingular.

- For convenience, we define $\nabla \times \mathbf{v}_1 = \hat{\mathbf{z}}\nabla^2\psi$, which yields

$$\psi(\mathbf{x}) = \sum_i n_i \log(|\mathbf{x} - \mathbf{x}_i|).$$

  Here, ψ is analogous to the potential due to point charges in two dimensions. (To use three-dimensional vector operations, we have implicitly extended $\mathbf{v}$ so that $\mathbf{v}(\mathbf{x}, z) = \mathbf{v}(\mathbf{x}, 0)$.) Note that there is an ambiguity in the definition of ψ, but it can be absorbed into that of ϕ.

- Using the curl of curl identity, we can solve for $\mathbf{v}_1$, giving

$$\mathbf{v} = \nabla\phi - \nabla \times (\hat{\mathbf{z}}\psi).$$

  To understand the second term, note that $\nabla \times (\hat{\mathbf{z}}\psi)$ is just $\nabla\psi$ rotated by $\pi/2$. This lets us convert between electric charge intuition ($\nabla\psi$), where field flows in/out and fluid flow intuition ($\nabla \times (\hat{\mathbf{z}}\psi)$), where fluid flows around.

- The energy of a configuration decomposes into three terms,

$$\beta H = \frac{K}{2}\int d^2\mathbf{x}\left[(\nabla\phi)^2 - 2\nabla\phi\cdot\nabla\times(\hat{\mathbf{z}}\psi) + (\nabla\times\hat{\mathbf{z}}\psi)^2\right].$$

  The first term is just the energy of topologically trivial spin waves. The second vanishes by integration by parts. The third term is equal to

$$(\nabla\times\hat{\mathbf{z}}\psi)^2 = (\nabla\psi)^2 \sim -\psi\nabla^2\psi$$

  where we integrated by parts in the last step. Since ψ is analogous to potential, this term is analogous to the E^2 field energy and causes an interaction between charges.

- Plugging in the explicit forms of ψ and $\nabla^2\psi$, the interaction is

$$\beta H_1 = -2\pi^2 K\sum_{i,j} n_i n_j C(\mathbf{x}_i - \mathbf{x}_j), \quad C(\mathbf{x}) = \log(|\mathbf{x}|)/2\pi.$$

  The energy diverges for $i = j$ because our approximation breaks down; in that case we just recover the core energy $\mathcal{E}^0$, so we really have

$$\beta H_1 = \sum_i \beta\mathcal{E}^0_{n_i} - 4\pi^2 K\sum_{i<j} n_i n_j C(\mathbf{x}_i - \mathbf{x}_j).$$

- At low temperature, we can factor the partition function as

$$Z = Z_{\text{sw}} Z_{\text{vortex}}$$

  where the terms represent spin waves and vortices. Since the spin wave contribution is nonsingular, the phase transition must be due to vortex unbinding.

- When integrating by parts, we dropped surface terms that contribute $(\sum_i n_i)(\log L)$ to the energy, like the divergent energy of a single vortex. Therefore at low temperature we only consider charge-neutral configurations, containing vortices of charge ± 1, as these have the lowest energy. Then

$$Z_{\text{vortex}} = \sum_{N=0}^{\infty} y_0^N \int\prod_{i=1}^{N} d\mathbf{x}_i \exp\left[4\pi^2 K\sum_{i<j} n_i n_j C(\mathbf{x}_i - \mathbf{x}_j)\right]$$

  where $y_0 = \exp(-\beta\mathcal{E}^0_{\pm 1})$ is the fugacity of vortices, $n_i = \pm 1$, and we sum over charge-neutral configurations where the vortices are separated by at least a distance a.

7.4 The Coulomb Gas

As shown earlier, the phase transition of the XY model is due to the unbinding of vortex pairs, and has a direct analogue to the Coulomb gas. At low temperatures, charges appear only in tightly-bound dipole pairs. At high temperatures, the charges dissociate to form a plasma.

- In the Coulomb gas, the phase transition is an insulator to metal transition. The bare Coulomb interaction between two external charges is $C(x)$. In the low-temperature phase, the internal charges act as a dielectric, reducing the interaction to $C(x)/\epsilon$. In the high-temperature phase, the internal charges screen the external ones, so $C(x)$ decays exponentially.

- We can compute the effective interaction $V(x)$ perturbatively in y_0, so the lowest-order term contains one internal dipole. Equivalently, we can calculate the dielectric constant ϵ.
- Suppose the Coulomb gas forms a square of side length L. If we put in an electric field E, the internal field E' satisfies
$$(E - E')L = q_{\text{in}}$$
by Gauss's law, where q_{in} is due to dipoles straddling the edge of the Gaussian surface.
- Integrating over possible dipole lengths and orientations,
$$q_{\text{in}} = \int_a^\infty \frac{r\,dr}{a^2} \int_0^{2\pi} d\theta\, \frac{Lr\cos\theta}{a^2}\,(2\pi)\, e^{KE'(2\pi r)\cos\theta} y_0^2.$$
The factor $Lr\cos\theta$ describes the possible positions of a dipole straddling the edge. The lengths are divided by a, because as justified earlier, we can only specify the dipole's position up to this resolution. The 2π factor is the charge, and the exponential factor is the Boltzmann weight describing dipole alignment, where $2\pi r$ is the dipole moment.
- Expanding the exponential to first order and performing the integrals shows
$$K_{\text{eff}} = K - 4\pi^3 K^2 y_0^2 a^{2\pi K - 4} \int_a^\infty dr\, r^{3-2\pi K} + O(y_0^4), \quad \epsilon = \frac{K}{K_{\text{eff}}}.$$
Since y_0 is small, this is a valid perturbative expansion as long as the integral converges, which occurs for $K > K_c = 2/\pi$. Below this critical coupling, each term diverges; the situation is analogous to computing fluctuations in Landau–Ginzburg theory for $d < 4$.
- Now we perform RG. The microscopic cutoff is the core size a, which affects the parameters K and y_0. When we change a to ba, we demand that the partition function
$$Z_1 = y_0(a) \left(\frac{L}{a}\right)^{2-\pi K}$$
remain the same, as well as the coupling K_{eff}. Working to lowest order in y_0 gives
$$\frac{dK^{-1}}{d\ell} = 4\pi^3 a^4 y_0^2 + O(y_0^4), \quad \frac{dy_0}{d\ell} = (2 - \pi K) y_0 + O(y_0^3).$$
By construction, K_{eff} is the value of the coupling K once we have RG flowed to $y_0 = 0$. Then the true XY model with coupling K is equivalent to an XY model with no topological defects and coupling K_{eff}.
- We have a line of fixed points for $y_0 = 0$, which are stable for $K_{\text{eff}} > 2/\pi$. These represent the insulating phase. For smaller couplings, the fixed points are unstable and we are pushed to the $K \to 0$, $y_0 \to \infty$ fixed point, representing the conducting phase.
- To understand the critical behavior, we expand near the critical point, defining
$$x = K^{-1} - \pi/2, \quad y = y_0 a^2$$
which gives, to lowest order,
$$\frac{dx}{d\ell} = 4\pi^3 y^2, \quad \frac{dy}{d\ell} = \frac{4xy}{\pi}.$$
We will derive these equations by another route below.

7.5 ∗ The Sine-Gordan Model

We can also compute the RG flow of the Coulomb gas by mapping it to the Sine-Gordan model.

- The Sine-Gordan model has

$$F = \int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2 - \lambda\cos(\beta\phi)$$

  where ϕ is a real scalar field and we work in $d = 2$. The model is interesting because it contains solitons, and it is named after Klein-Gordan theory.

- We claim the Sine-Gordan model is equivalent to the Coulomb gas. To see this, consider discretizing space into lattice points $\mathbf{X}_\alpha$ separated by a, and write a Coulomb gas configuration as an assignment $V_\alpha \in \{-1, 0, 1\}$ reflecting the presence of absence of an (anti)-vortex. Then

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \exp\left(\frac{\pi}{e^2}\sum_{\alpha\neq\beta} V_\alpha V_\beta \log\left(\frac{|\mathbf{X}_\alpha - \mathbf{X}_\beta|}{a}\right) - \sum_\alpha V_\alpha^2 F_{\text{core}}\right).$$

  We restrict to neutral configurations, so $\sum_\alpha V_\alpha = 0$.

- Next, we use the fact that the logarithm is the Green's function for the two-dimensional Laplacian, so by our path integral computations,

$$\int \mathcal{D}\phi\, \exp\left(-\int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2 + f(\mathbf{x})\phi(\mathbf{x})\right) \sim \exp\left(-\frac{1}{4\pi}\int d\mathbf{x}d\mathbf{y}\, f(\mathbf{x})\log|\mathbf{x}-\mathbf{y}|f(\mathbf{y})\right)$$

  where we omit a constant normalization factor. Then the partition function becomes

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \int \mathcal{D}\phi\, \exp\left(-\int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2 + \sum_\alpha \frac{2\pi i}{e} V_\alpha \phi(\mathbf{X}_\alpha) - V_\alpha^2 F_{\text{core}}\right).$$

  That is, the vortices couple to the field ϕ, but with an imaginary energy.

- Finally, performing the sum over $\{V_\alpha\}$, the interaction factors become

$$\prod_\alpha \left(1 + 2e^{-F_{\text{core}}}\cos(2\pi\phi_\alpha/e)\right) \sim \exp\left(\frac{2}{a^2}e^{-F_{\text{core}}}\int d\mathbf{x}\, \cos\left(\frac{2\pi\phi}{e}\right)\right)$$

  so that we arrive at the Sine-Gordan model with correspondences

$$\lambda = \frac{2e^{-F_{\text{core}}}}{a^2}, \quad \beta = \frac{2\pi}{e}.$$

  The cutoff a is to be interpreted as the cutoff scale.

Now we proceed to compute the RG flow at first order.

- As usual, we split the field into high and low Fourier components $\phi_{\mathbf{k}} = \phi^-_{\mathbf{k}} + \phi^+_{\mathbf{k}}$ above and below Λ/ζ, and integrating out the high momentum modes gives the effective free energy

$$F'[\phi^-] = F_0[\phi^-] - \log\langle e^{-F_I[\phi^- + \phi^+]}\rangle_+, \quad F_0[\phi] = \int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2, \quad F_I[\phi] = -\lambda_0\int d\mathbf{x}\, \cos(\beta_0\phi)$$

  where the expectation value is taken in the free ensemble for the fast modes. We have added subscripts to the coupling constants λ and β since they are the values before renormalization.

- The parameter β_0 is not necessarily small, so we can't expand the cosine in a series. Hence diagrammatics don't make sense since have an infinite number of vertices. Instead, we expand in the small parameter λ_0, reflecting the fact that a single vortex has a high energy cost.

- Since λ_0 is small, we expand the exponential to lowest order,

$$\log\langle e^{-F_I[\phi^- + \phi^+]}\rangle_+ \approx \lambda_0 \int d\mathbf{x}\, \langle \cos(\beta(\phi^- + \phi^+))\rangle_+ = \frac{\lambda_0}{2}\int d\mathbf{x} \sum_{\sigma=\pm 1} e^{i\beta\sigma\phi^-}\langle e^{i\beta\sigma\phi^+}\rangle_+.$$

  Now we need to evaluate the expectation value of an exponential in a Gaussian ensemble, which is done precisely by the Wick identity,

$$\langle e^{i\beta\sigma\phi^+(\mathbf{x})}\rangle_+ = e^{-(\beta_0^2/2)\langle\phi^+(\mathbf{x})\phi^+(\mathbf{x})\rangle_+}.$$

  The propagator can be evaluated explicitly as

$$\langle\phi^+(0)\phi^+(0)\rangle_+ = \int_{\Lambda/\zeta}^{\Lambda} \frac{d\mathbf{k}}{k^2} = \frac{\log\zeta}{2\pi}.$$

- Combining our results and performing the sum over σ, we find a renormalization of λ,

$$\lambda' = \lambda_0 \zeta^{-\beta^2/4\pi}.$$

  We perform the usual rescaling $\mathbf{x} \to \mathbf{x}/\zeta$. No rescaling of the field is necessary, and we find

$$\lambda(\zeta) = \lambda_0 e^{2-\beta^2/4\pi}.$$

- For $\beta^2 > 8\pi$ the RG flow reduces λ, so the coupling is irrelevant and we flow to a free field theory. For $\beta^2 < 8\pi$, the RG flow increases λ. In the limit of high λ, fluctuations of the field are suppressed, so we can expand the cosine to get a mass term $\lambda_0\beta^2\phi^2/2$ and hence a finite correlation length. Then we predict a phase transition at $\beta^2 = 8\pi$.

- This is equivalent to $e^2 = \pi/2$, which is remarkably exactly the same value we found by heuristic arguments earlier. At this phase transition, we have the universal result $\eta = 1/4$.

Next, we give the results at second order.

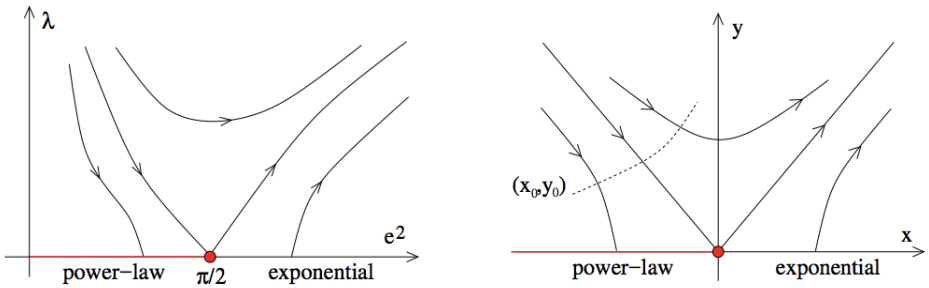
- At second order, we generate the term $\cos(2\beta_0\phi)$, reflecting the presence of two vortices at a lattice point, along with a renormalization of the kinetic term.

- Ignoring the new term, the RG flow equations are

$$\frac{d\lambda}{ds} = \left(2 - \frac{\pi}{e^2}\right)\lambda, \quad \frac{de^2}{ds} = 8\pi^2 C\lambda^2 e^2$$

  where C is a numerical constant we will not compute. Since the interaction energy of vortices is proportional to $1/e^2$, RG flow weakens the interaction, physically because of screening by vortex-antivortex pairs.

- The structure of the RG flow is shown below.



  If λ starts very low, then our first-order results apply: we end up in the low temperature phase if $e^2 < \pi/2$. For finite λ, we require a slightly lower value of e^2.

- We can 'zoom in' around the phase transition, defining

$$x = e^2 - \frac{\pi}{2}, \quad y = \sqrt{8\pi^2 C}\lambda, \quad \frac{dx}{ds} = y^2, \quad \frac{dy}{ds} \approx \frac{4}{\pi}xy.$$

  The RG flows are hyperbola, with $x^2 - \pi y^2 = J$ conserved and the phase transition at $J = 0$.

- As we vary the temperature for a fixed system, both e^2 and λ are changed. Therefore we can think of J as a function of T, with leading behavior

$$J(T) \sim T_c - T.$$

  This allows us to infer some temperature dependence.

- Suppose we start with small $J < 0$. Then we find

$$s = \frac{\pi}{\sqrt{|J|}}\left(\tan^{-1}\left(\frac{x}{\sqrt{|J|}}\right) - \tan^{-1}\left(\frac{x_0}{\sqrt{|J|}}\right)\right).$$

  When $x \sim 1$, we have $y \sim 1$, so vortices are common, so the correlation length is $\xi \sim a$. Working backwards, and dropping coefficients, that means

$$\xi \sim ae^s \sim \exp\left(\frac{1}{\sqrt{|J|}}\right) \sim \exp\left(\frac{1}{\sqrt{T - T_c}}\right)$$

  which is an extremely strong divergence, faster than any power.

- Applying hyperscaling, $F \sim (L/\xi)^2$ so that

$$F \sim \frac{1}{\xi^2} \sim \exp\left(-\frac{1}{\sqrt{T - T_c}}\right).$$

  This is an extremely weak singularity, weaker than any power. In terms of Ehrenfest's classification, the Kosterlitz–Thouless transition is of infinite order.

- Just below the transition, the quantity $1/e^2$ flows to $2/\pi$, while just above it, it flows to zero. Then we predict a discontinuity of $2/\pi$ in $1/e^2$ at the transition, a universal result, with a subleading square root singularity in T. This has been confirmed in superfluid helium.

The Kosterlitz–Thouless transition can also be used to understand melting.

- Solids spontaneously break translational symmetry, which is regained when the solid melts. However, by the Mermin–Wagner theorem, translational symmetry can't be broken in the first place in $d \le 2$, which means we should not see a sharp melting phase transition. Despite this, it is observed in $d = 2$.
- More quantitatively, fixing a crystal lattice and writing the distortion field as $\mathbf{u}(\mathbf{x})$, we have

  $$\langle u_i(\mathbf{q}) u_j(\mathbf{q}') \rangle \sim \frac{\not\delta(\mathbf{q}+\mathbf{q}')}{q^2} T_{ij}$$

  for some tensor T_{ij} which depends on the crystal lattice, so

  $$\langle (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{0}))^2 \rangle \sim \int đ\mathbf{q} \frac{2 - 2\cos(\mathbf{q}\cdot\mathbf{x})}{q^2}.$$

  This integral diverges for large $\mathbf{x}$ for $d \le 2$. The order parameter can be taken to be $e^{i\mathbf{G}\cdot\mathbf{u}(\mathbf{x})}$ for any $\mathbf{G}$ in the reciprocal lattice, and its correlators decay algebraically for $d = 2$.
- We may also define an order parameter for orientational order,

  $$\Psi(\mathbf{x}) = e^{in\theta(\mathbf{x})}, \quad \theta(\mathbf{x}) = -\frac{1}{2}\hat{z}\cdot\nabla\times\mathbf{u}(\mathbf{x})$$

  where $n = 6$ for a triangular lattice and $n = 4$ for a square lattice.
- Unlike translational order, there is long-range orientational order. This is rather subtle; it doesn't contradict the Mermin–Wagner theorem because soft phonons mediate a long-range interaction between orientations at different lattice points. A related subtlety is that solids don't have any low-energy excitations corresponding to orientational deformations, since at long distances these "tear up the lattice".
- In the KTHNY theory, a two-dimensional solid is disordered by topological defects called dislocations, which result from extra or missing half lattice planes; they are associated with $\pi_1(T^2)$. There is a Kosterlitz–Thouless-style phase transition where the dislocations unbind, above which the orientational order decays algebraically.
- This "hexatic" phase is not a liquid. Instead, there is a further phase transition involving the unbinding of a new set of topological defects called disinclinations, which are very similar to the vortices in the XY model.