Lecture Notes on

Supersymmetry

Kevin Zhou

kzhou7@gmail.com

These notes cover supersymmetry, closely following the Part III and MMathPhys Supersymmetry courses as lectured in 2017/2018 and 2018/2019, respectively. The primary sources were:

- Fernando Quevedo's Supersymmetry lecture notes. A short, clear introduction to supersymmetry covering the topics required to formulate the MSSM. Also see Ben Allanach's revised version, which places slightly more emphasis on MSSM phenomenology.
- Cyril Closset's Supersymmetry lecture notes. A comprehensive set of supersymmetry lecture notes with more emphasis on theoretical applications. Contains coverage of higher SUSY, and spinors in various dimensions, the dynamics of 4D $\mathcal{N} = 1$ gauge theories, and even a brief overview of supergravity.
- Aitchison, *Supersymmetry in Particle Physics*. A friendly introductory book that covers the basics with a minimum of formalism; for instance, the Wess–Zumino model is introduced without using superfields. Also gives an extensive treatment of subtleties in two-component spinor notation. The last half covers the phenomenology of the MSSM.
- Wess and Bagger, *Supersymmetry and Supergravity*. An incredibly terse book that serves as a useful reference. Most modern sources follow the conventions set here. Many pages consist entirely of equations, with no words.

We will use the conventions of Quevedo's lecture notes. As such, the metric is mostly negative, and a few other signs are flipped with respect to Wess and Bagger's conventions. The most recent version is here; please report any errors found to kzhou7@gmail.com.

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1 Introduction

1.1 Motivation

We begin with a review of the Standard Model and its problems.

- A spacetime symmetry is one that acts explicitly on the spacetime coordinates,

$$x^\mu \to x'^\mu(x^\nu)$$

  and include Poincare transformations in special relativity, and more generally, general coordinate transformations in general relativity.

- An internal symmetry corresponds to transformations of the different fields in a field theory,

$$\Phi^a(x) \to M^a{}_b \Phi^b(x).$$

  If M is constant, the symmetry is global, and if $M = M(x)$ it is local.

- Symmetries constrain the interactions between fields. For example, most quantum field theories of vector bosons are non-renormalizable, but gauge theories are renormalizable.

- Symmetries may also be spontaneously broken. This is important phenomenologically because it naturally introduces an energy scale in the system, determined by the VEV, and allows for more complex fundamental symmetries than we observe at low energies.

- The SM has Poincare symmetry and a gauge $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry, spontaneously broken to $SU(3)_C \times U(1)_A$.

- The hierarchy problem is the result

$$\frac{m_h}{M_p} \sim 10^{-17}$$

  which is technically unnatural; there is nothing protecting m_h from receiving $O(M_p)$ quantum corrections. Similarly, the cosmological constant problem is

$$(\Lambda/M_p)^4 \sim 10^{-120}.$$

  A related issue is how the 20 free parameters of the SM are determined. Finally, the SM does not account for dark matter.

Next, we turn to historical motivations for supersymmetry.

- In the 1960's, much progress was made by classifying hadrons into multiplets, and there were attempts to enlarge the symmetry groups by including spacetime symmetries.

- The Coleman–Mandula theorem (1967) states that spacetime and internal symmetries cannot be combined nontrivially in a relativistic theory with nontrivial scattering, a mass gap, and finitely many particles. More precisely, the symmetry group of the S-matrix must be a direct product of the Poincare group and an internal symmetry group. (Conformal field theories (CFTs) evade this theorem because they don't have a mass gap, allowing the larger spacetime symmetry group $SO(2, d)$ in d spacetime dimensions.)

- In 1971, Gelfand and Likhtman extended the Poincare algebra by adding generators that transformed like spinors and satisfied anticommutation relations, thus inventing SUSY; this evaded the Coleman–Mandula theorem because the symmetry was described by a Lie superalgebra rather than a Lie algebra. Note that the spin-statistics theorem ensures that in all dimensions, SUSY generators must be spinors.
- Simultaneously, Ramond, Neveu, and Schwarz found that string theory extended with fermions was a two-dimensional supersymmetric theory on the worldsheet, inventing superstring theory. The string worldsheet also has conformal symmetry, making it a superconformal field theory (SCFT).
- In the 1970's, neutrinos were thought to be massless. In 1973, Volkov and Akulov proposed that neutrinos were Goldstone *fermions*, called Goldstinos, due to the spontaneous breaking of SUSY.
- In 1974, Wess and Zumino wrote down the first example of an interacting four-dimensional quantum field theory with linearly realized SUSY. Simultaneously, Salam and Strathdee invented the tools of superfields and superspace, coining the term 'supersymmetry'.
- In 1975, Haag, Lopuszanski, and Sohnius generalized the Coleman–Mandula theorem to essentially state that the most general symmetry possible was a direct product of the super Poincare group and internal symmetries.
- Making Poincare symmetry local yields general coordinate transformations and hence general relativity. In 1976, Friedman, van Niewenhuizen, and Ferrara, and Deser and Zumino made SUSY local, yielding supergravity. The superpartner of the graviton was the spin $3/2$ gravitino.
- From 1977 to the 1980's, SUSY phenomenology was developed. It was demonstrated that SUSY could solve the hierarchy problem in a natural way, though this is less relevant today.
- Simultaneously, in 1977 Gliozzi, Scherk, and Olive demonstrated how to remove the tachyon from the Ramond-Neveu-Schwarz model, and conjectured the resulting theory had *spacetime* supersymmetry. From 1981 to 1984, Green and Schwarz proved this conjecture, discovering an anomaly cancellation mechanism for superstring theory in $d = 10$ and starting the first superstring revolution.
- In 1991, LEP performed precision tests of the SM. It was found that gauge coupling unification did not occur for the SM, presenting problems for GUTs, but would happen for the MSSM as long as superpartners had masses in the range $100\,\text{GeV}$ to $10\,\text{TeV}$.
- In 1994, Seiberg and Witten investigated $\mathcal{N} = 2$ superstring theory nonperturbatively, discovering M-theory and starting the second superstring revolution.
- In 1996, Strominger and Vafa counted the microstates of a black hole in superstring theory to confirm the Bekenstein-Hawking formula $S = A/4$.
- In 1998, the AdS/CFT duality was proposed by Maldacena, showing that certain CFTs in d dimensions are dual to quantum gravity theories in AdS space in $d + 1$ dimensions. The best studied instances of the AdS/CFT duality involve SCFTs which are dual to superstring theories in AdS, making supersymmetry a useful tool for studying quantum gravity.

Next, we discuss the hierarchy problem in more detail.

- Consider a Higgs potential of the form
$$V = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{4}(\phi^\dagger \phi)^2.$$
Then the Higgs vev, which sets the weak scale, is
$$\langle \phi \rangle = \sqrt{2}\mu/\sqrt{\lambda}.$$
For perturbation theory to apply, λ should not be too large, so $\mu \lesssim \langle \phi \rangle$.

- The issue is not that μ is small, but that quantum effects give large corrections to μ. This doesn't happen for gauge boson masses, which are held at zero by gauge symmetry, or for spinor masses, because chiral symmetry is restored when the mass vanishes. Then $\delta m \sim m \log \Lambda$, which is reasonably small even when Λ is the Planck scale.

- On the other hand, the one-loop contribution due to the Higgs is
$$\delta \mu^2 \sim \lambda \int^\Lambda \frac{d^4 k}{k^2 - M_H^2} \sim \lambda \Lambda^2$$
so if Λ is the Planck scale, μ^2 must be fine-tuned to get an acceptable observed value of μ^2_{phys}, the coefficient of $\phi^\dagger \phi$ in the 1PI effective action. Thus to avoid fine tuning there must be new physics around the TeV scale.

- One solution is to postulate that spontaneous symmetry breaking occurs 'dynamically'. In a technicolor theory, the Higgs is a composite of fermions, analogous to Cooper pairs in BCS theory, so the theory above is only an effective field theory valid up to the TeV scale. However, this theory has issues with giving masses to fermions.

- Another, more radical solution is to set the Planck scale to the TeV scale; this is consistent if there are large extra dimensions. We'll put these ideas aside and focus on SUSY.

- The one-loop contribution to $\delta\mu^2$ above can also be canceled by fermion contributions. Consider a fermion with Yukawa coupling g_f to the Higgs. Then
$$\delta \mu^2 \sim -g_f^2 \int^\Lambda d^4 k \; \mathrm{tr} \, \frac{1}{(\not{k} - m_f)^2} \sim -4 g_f^2 \Lambda^2$$
where the minus sign comes from the fermion loop. Then the quadratic divergence cancels if, for every boson, there is a fermion whose coupling to the Higgs is related; this is guaranteed by SUSY.

- Even given this cancellation, there is still a logarithmic divergence,
$$\delta \mu^2 \sim \lambda (M_H^2 - m_f^2) \log \Lambda$$
where we dropped all numerical factors, which depends quadratically on the particle masses. This is completely generic; we would even get a contribution of m^2 from a particle of mass m that didn't couple directly to the Higgs, by a multi-loop diagram. Hence the parameter μ^2 is quadratically sensitive to any scale associated with new physics.

- Thus to avoid fine tuning, new physics should arise at the TeV scale. Moreover, if that physics is SUSY, the superpartner masses should generally be around the TeV scale. This is especially important in SUSY GUTs, where there are many very heavy particles.
- In the MSSM, applying naturalness to the coefficients shows that the Higgs should be no heavier than 140 GeV. By contrast, in the SM there is no constraint, unless we count perturbative unitarity, which bounds the Higgs mass by a few hundred GeV.

Note. A cartoon explanation of the Coleman–Mandula theorem. Essentially, the theorem states that conserved charges from internal symmetries can't have Lorentz indices; the only such charges are momentum P_μ and angular momentum $M_{\mu\nu}$ which arise from spacetime symmetries. Suppose we had another such charge $Q_{\mu\nu}$. By Lorentz invariance,

$$Q_{\mu\nu}|p\rangle = (\alpha p_\mu p_\nu + \beta g_{\mu\nu})|p\rangle.$$

Now consider a two-particle state. We suppose that $Q_{\mu\nu}$ values are additive, conserved, and act on only one particle at a time. Then

$$Q_{\mu\nu}|p^{(1)}, p^{(2)}\rangle = (\alpha(p_\mu^{(1)} p_\nu^{(1)} + p_\mu^{(2)} p_\nu^{(2)}) + 2\beta g_{\mu\nu})|p^{(1)}, p^{(2)}\rangle.$$

Therefore, in a $1 + 2 \to 3 + 4$ scattering process we have

$$p^{(1)} + p^{(2)} = p^{(3)} + p^{(4)}, \quad p_\mu^{(1)} p_\nu^{(1)} + p_\mu^{(2)} p_\nu^{(2)} = p_\mu^{(3)} p_\nu^{(3)} + p_\mu^{(4)} p_\nu^{(4)}.$$

However, these conditions are so restrictive that there are no nontrivial solutions! We can only have forward or backwards scattering.

Note. A preview of the SUSY algebra. We will have a spinorial generator Q_a with $a = 1, 2$, which relates bosons and fermions; then the above argument fails at the first step, since we cannot superpose the two. The Q_a satisfy anticommutation relations among themselves, and commute with H. Then we have

$$[\{Q_a, Q_b\}, H] = 0.$$

Now, $\{Q_a, Q_b\}$ should be a "spin one' object', so in a relativistic field theory it should be a four-vector. The only conserved four-vector is P_μ, so

$$\{Q_a, Q_b\} \sim P_\mu.$$

Hence supersymmetry transformations inevitably relate internal and spacetime symmetries. They function as a kind of "square root" of translations, and hence take us from ordinary space to superspace like how $\sqrt{-1}$ takes us from the real line to the complex plane. Note that the spin 1/2 generator Q_a is the *only* exception to the Coleman–Mandula theorem, i.e. the Haag–Lopuszanski–Sohnius rules out spin 3/2 generators $Q_{\mu a}$, and so on.

1.2 The Poincare Group

We begin by reviewing the Poincare algebra.

- The Poincare group corresponds to the basic symmetries of special relativity. It acts on spacetime coordinates by

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\mu + a^\mu$$

where the Lorentz transformations Λ satisfy

$$\Lambda^T \eta \Lambda = \eta, \quad \eta = \mathrm{diag}(1, -1, -1, -1).$$

We will focus on the proper orthochronous Lorentz group $SO(1,3)^\uparrow$, while the full Lorentz group is $O(3,1) = \{1, \Lambda_P, \Lambda_T, \Lambda_{PT}\} \times SO(1,3)^\uparrow$. Below we'll just write $SO(1,3)$ for $SO(1,3)^\uparrow$.

- Infinitesimally, we have

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu, \quad a^\mu = \epsilon^\mu$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$. If the Poincare group is represented by $U(\Lambda, a)$ on a Hilbert space, then infinitesimally we define

$$U(1+\omega, \epsilon) = 1 - \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} + i\epsilon_\mu P^\mu.$$

A useful explicit expression is

$$(M^{\rho\sigma})^\mu{}_\nu = -i(\eta^{\mu\sigma}\delta^\rho_\nu - \eta^{\rho\mu}\delta^\sigma_\nu).$$

Note that we use the same notation for the abstract Poincare algebra elements and their representations on a Hilbert space, since we will use the latter constantly. By the definition of a representation, the commutator on the latter is the bracket on the former.

- We now find the Poincare algebra. Since translations commute in the Hilbert space, we have

$$[P_\mu, P_\nu] = 0.$$

- Since P^μ is a vector, it transforms under the Lorentz group as

$$P^\sigma \to \Lambda^\sigma{}_\rho P^\rho = P^\sigma + \frac{1}{2}\omega_{\alpha\rho}(\eta^{\sigma\alpha}P^\rho - \eta^{\sigma\rho}P^\alpha).$$

On the other hand, we can compute the transformation of P^μ explicitly in a representation of the Poincare group on a Hilbert space, where the operator P^μ transforms as

$$P^\sigma \to U^\dagger P^\sigma U = \left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) P^\sigma \left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = P^\sigma - \frac{i}{2}\omega_{\mu\nu}(P^\sigma M^{\mu\nu} - M^{\mu\nu}P^\sigma)$$

from which we read off the commutation relation

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma})$$

- By similar reasoning, we have

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}).$$

As above, this just states the tensorial transformation properties of $M^{\mu\nu}$ infinitesimally.

- We define the Hermitian and anti-Hermitian generators of rotations and boosts by

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \quad K_i = M_{0i}$$

where $\epsilon_{123} = \epsilon^{123} = 1$. Note that the indices on M here are *not* lowered by the metric, $M_{0i} = M^{0i}$. The Lorentz algebra is

$$[K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [J_i, J_j] = i\epsilon_{ijk}J_k.$$

- We now define the linear combinations

$$A_i = \frac{1}{2}(J_i + iK_i), \quad B_i = \frac{1}{2}(J_i - iK_i)$$

which satisfy the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ commutation relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0.$$

Hence we conclude $\mathfrak{so}(3,1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as complex Lie algebras. Note that $\mathbf{J} = \mathbf{A} + \mathbf{B}$, and under parity $\mathbf{J} \to \mathbf{J}$ and $\mathbf{K} \to \mathbf{K}$, so $\mathbf{A}$ and $\mathbf{B}$ are interchanged. This leads to the usual classification of representations of the Lorentz group.

- However, there is another route. There is a homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$ as $SL(2,\mathbb{C})$ is the universal/double cover of $SO(1,3)$. For a four-vector X, we define

$$X = x_\mu e^\mu = (x_0, x_1, x_2, x_3), \quad \tilde{x} = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad \sigma^\mu = (1, \boldsymbol{\sigma}).$$

However, if we use mostly positive signature, we instead must define $\sigma^\mu = (-1, \boldsymbol{\sigma})$.

- Now, $SO(1,3)$ and $SL(2,\mathbb{C})$ act on these spaces by

$$X \to \Lambda X, \quad \tilde{x} \to N\tilde{x}N^\dagger, \quad \Lambda \in SO(1,3), \quad N \in SL(2,\mathbb{C})$$

so we can construct the homomorphism by mapping back and forth. Explicitly, it is

$$\Lambda^\mu{}_\nu = \frac{1}{2}\operatorname{tr}\overline{\sigma}^\mu N \sigma_\nu N^\dagger.$$

- This map is well-defined and surjective since the only constraint on the $\tilde{x}$ transformations is

$$\det \tilde{x} = x_0^1 - x_1^2 - x_2^2 - x_3^2 = \text{constant}$$

while the only constraint on the Lorentz transformations is

$$|X|^2 = x_0^1 - x_1^2 - x_2^2 - x_3^2 = \text{constant}.$$

It is a double cover since both $N = \pm 1$ correspond to $\Lambda = 1$.

Note. The topology of $SL(2,\mathbb{C})$. To see it, use the polar decomposition

$$N = e^H U$$

where H is Hermitian and U is unitary. We may parametrize them as

$$H = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix}, \quad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}$$

where $x^2 + y^2 + z^2 + w^2 = 1$. The set of H is $\mathbb{R}^3$ while the set of U is S^3, so $SL(2,\mathbb{C}) \cong \mathbb{R}^3 \times S^3$, while the Lorentz group mods out S^3 by $\mathbb{Z}_2$.

1.3 Spinors in Four Dimensions

As shown above, we can use the representation theory of $SL(2,\mathbb{C})$ to find the projective representations of $SO(1,3)$. This is especially useful for the fundamental spinor representations. Here we will use the tensor methods explained in the notes on Group Theory.

- The fundamental representation transforms as
$$\psi_\alpha \to N_\alpha{}^\beta \psi_\beta$$
and contains left-handed Weyl spinors. The conjugate representation
$$\overline{\chi}_{\dot\alpha} \to N^*{}_{\dot\alpha}{}^{\dot\beta}\,\overline{\chi}_{\dot\beta}$$
contains right-handed Weyl spinors, where N^* is the conjugate of N.

- These representations also have dual/contravariant representations
$$\psi^\alpha \to \psi^\beta (N^{-1})_\beta{}^\alpha, \quad \overline{\chi}^{\dot\alpha} \to \overline{\chi}^{\dot\beta}(N^{*^{-1}})_{\dot\beta}{}^{\dot\alpha}.$$

- We are using a redundant notation: the ψ and χ don't matter, but dotted indices are associated with bars. This is useful because we can then write expressions unambiguously without indices.

- For the matrices N, dotted indices always accompany a conjugate, so they're redundant as we always write the conjugate explicitly. We simply assign indices to N so that the indices match up properly; note that the first index is always down.

- The invariant tensors in $SL(2,\mathbb{C})$ are delta functions δ^α_β and $\delta^{\dot\alpha}_{\dot\beta}$ and the Levi–Civita symbols
$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta}, \quad \epsilon^{12} = 1$$
where the minus sign ensures $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^\alpha_\gamma$. They are invariant because
$$\epsilon_{\alpha\beta} \to N_\alpha{}^\rho N_\beta{}^\sigma \epsilon_{\rho\sigma} = \epsilon_{\alpha\beta}\,\det N = \epsilon_{\alpha\beta}$$
with similar proofs for the others. Then the Levi–Civita can be used to invert matrices,
$$\epsilon^{\sigma\delta} N_\delta{}^\beta \epsilon_{\beta\alpha} = (N^{-1})_\alpha{}^\sigma$$

- The Levi–Civitas can be used to raise or lower indices. This is a bit tricky because $\epsilon^{\alpha\beta}$ is not symmetric; by convention we always contract the second index. We define
$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \overline{\chi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\overline{\chi}_{\dot\beta}, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \overline{\chi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\chi}^{\dot\beta}.$$
All these objects transform as their index placement would suggest.

- We can also compute a transformation for the Pauli matrices $(\sigma^\mu)_{\alpha\dot\alpha}$ which have mixed indices. The equation $\tilde{x} \to N\tilde{x}N^\dagger$ implies $\tilde{x}$ has one dotted and one undotted index, so
$$\tilde{x} = (x_\mu \sigma^\mu)_{\alpha\dot\alpha} \to N_\alpha{}^\beta (x_\nu\sigma^\nu)_{\beta\dot\gamma} N^*{}_{\dot\alpha}{}^{\dot\gamma} = \Lambda_\mu{}^\nu x_\nu (\sigma^\mu)_{\alpha\dot\alpha}$$
which gives the transformation rule
$$(\sigma^\mu)_{\alpha\dot\alpha} = N_\alpha{}^\beta N^*{}_{\dot\alpha}{}^{\dot\gamma} \Lambda^\mu{}_\nu\,(\sigma^\nu)_{\beta\dot\gamma}$$
which is exactly what we would expect from the index structure.

- It is also useful to define

$$(\overline{\sigma}^\mu)^{\dot\alpha\alpha} \equiv (\sigma^\mu)^{\alpha\dot\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma^\mu)_{\beta\dot\beta} = (1, -\boldsymbol{\sigma})$$

  which obeys a similar transformation law; note that if we didn't 'swap the indices' then the matrix $\overline{\sigma}^2$ would have the wrong sign.

- There are some useful identities for σ and $\overline{\sigma}$. They form a Clifford algebra, as

$$(\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu)_\alpha^{\ \beta} = 2\eta^{\mu\nu}\delta_\alpha^\beta.$$

  We may think of $\sigma^\mu_{\alpha\dot\alpha}$ as a set of Clebsch–Gordan coefficients for the identity $(1/2, 0)\times(0, 1/2) = (1/2, 1/2)$. The completeness of both bases is expressed by

$$(\sigma^\mu)_{\alpha\dot\beta}(\overline{\sigma}_\mu)^{\dot\gamma\delta} = 2\delta_\alpha^\delta\delta_{\dot\beta}^{\dot\gamma}, \quad \operatorname{tr}\sigma^\mu\overline{\sigma}^\nu = 2\eta^{\mu\nu}.$$

  Specifically, we can swap back and forth as

$$V_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}V_\mu \quad \leftrightarrow \quad V^\mu = \frac{1}{2}(\overline{\sigma}^\mu)^{\dot\alpha\alpha}V_{\alpha\dot\alpha}.$$

  Here, two irreps can multiply to another irrep because the Lorentz group is not semi-simple.

Next, we construct the generators of $SL(2, \mathbb{C})$ for the spinor representations.

- Just as the Dirac spinor is built from the Clifford algebra of gamma matrices, we have

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)_\alpha^{\ \beta}, \quad (\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)^{\dot\alpha}_{\ \dot\beta}.$$

  Then the matrices $\sigma^{\mu\nu}$, and the matrices $\overline{\sigma}^{\mu\nu}$, satisfy the Lorentz algebra,

$$[\sigma^{\mu\nu}, \sigma^{\lambda\rho}] = i(\eta^{\mu\rho}\sigma^{\nu\lambda} + \eta^{\nu\lambda}\sigma^{\mu\rho} - \eta^{\mu\lambda}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\lambda}).$$

  They obey the identity

$$\operatorname{tr}\,\sigma^{\mu\nu}\sigma^{\kappa\tau} = \frac{1}{2}(\eta^{\mu\kappa}\eta^{\nu\tau} - \eta^{\mu\tau}\eta^{\nu\kappa} + i\epsilon^{\mu\nu\kappa\tau}).$$

- The $(\sigma^{\mu\nu})_\alpha^{\ \beta}$ can also be used to project out the $(1, 0)$ representation in the product

$$(1/2, 1/2)\times(1/2, 1/2) = (1, 1) + (1, 0) + (0, 1) + (0, 0).$$

  That is, $V_\mu W_\nu(\sigma^{\mu\nu})_\alpha^{\ \beta}$ transforms in the $(1, 0)$. On the other hand, we also know that $(1, 0)$ is the symmetric product of two $(1/2, 0)$'s, which implies $(\sigma^{\mu\nu})^{\beta\alpha}$ is symmetric in α and β. Similarly, $(\overline{\sigma}^{\mu\nu})^{\dot\beta}_{\ \dot\alpha}$ projects out $(0, 1)$.

- One can show that the left-handed and right-handed spinors transform as

$$\psi_\alpha \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta}\psi_\beta, \quad \overline{\chi}^{\dot\alpha} \to \overline{\chi}^{\dot\beta}\exp\left(-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\right).$$

  In terms of the usual classification of Lorentz irreps we can show

$$\psi_\alpha\colon (A, B) = (1/2, 0), \quad J_i = \frac{1}{2}\sigma_i, \quad K_i = -\frac{i}{2}\sigma_i,$$
$$\overline{\chi}^{\dot\alpha}\colon (A, B) = (0, 1/2), \quad J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i.$$

- We also have the self-duality and anti self-duality identities

$$\sigma^{\mu\nu} = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, \quad \overline{\sigma}^{\mu\nu} = -\frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\overline{\sigma}_{\rho\sigma}.$$

  This ensures the transformations above are specified by 3 complex parameters, not 6. Here we define $\epsilon_{0123} = 1$, $\epsilon^{0123} = -1$ as is natural in general relativity, i.e. we use the opposite sign convention to $SL(2, \mathbb{C})$.

Next, we show how to multiply Weyl spinors.

- The contraction of Weyl spinors requires an ordering convention because the Levi–Civita is antisymmetric. As motivated below, we define

$$\chi\psi \equiv \chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha, \quad \overline{\chi}\overline{\psi} \equiv \overline{\chi}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} = -\overline{\chi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}}.$$

  That is, indices contract $\searrow$ for undotted indices and $\nearrow$ for dotted indices.

- In particular, we have

$$\psi\psi = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_2\psi_1 - \psi_1\psi_2.$$

  This appears to vanish classically, but since spinors are inherently anticommuting we choose to represent them as Grassmann numbers classically. Then

$$\psi\psi = 2\psi_2\psi_1, \quad \psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\psi).$$

  This also implies that contraction is symmetric, $\chi\psi = \psi\chi$ and $\overline{\chi}\overline{\psi} = \overline{\psi}\overline{\chi}$, and

$$(\theta\chi)(\theta\xi) = -\frac{1}{2}(\theta\theta)(\chi\xi), \quad (\overline{\theta}\overline{\chi})(\overline{\theta}\overline{\xi}) = -\frac{1}{2}(\overline{\theta}\overline{\theta})(\overline{\chi}\overline{\xi})$$

- One can conjugate a representation by just conjugating the vectors. That is, we define

$$\overline{\psi}_{\dot{\alpha}} = \psi_\alpha^\dagger, \quad \overline{\psi}^{\dot{\alpha}} = \psi^{\alpha\dagger}$$

  where the dagger simply stands for complex conjugation. Complex conjugation is defined to reverse the order of Grassmann numbers, $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$, which implies

$$(\chi\psi)^\dagger = \overline{\psi}\overline{\chi} = \overline{\chi}\overline{\psi}, \quad (\psi\sigma^\mu\overline{\chi})^\dagger = \chi\sigma^\mu\overline{\psi}$$

  where we used $((\sigma^\mu)_{\alpha\dot{\beta}})^* = ((\sigma^\mu)_{\alpha\dot{\beta}})^T = (\sigma^\mu)_{\beta\dot{\alpha}}$ since the σ^μ are Hermitian.

- Two-component spinor notation can be used to deal with tensor products of Lorentz representations. For example, we have the Fierz identity

$$\psi_\alpha\overline{\chi}_{\dot{\alpha}} = \frac{1}{2}(\psi\sigma_\mu\overline{\chi})\sigma^\mu_{\alpha\dot{\alpha}}, \quad (1/2, 0) \times (0, 1/2) = (1/2, 1/2)$$

  showing that a left-handed and right-handed spinor yield a vector $\psi\sigma_\mu\overline{\chi}$.

- Defining $(\sigma^{\mu\nu})_\alpha{}^\gamma \epsilon_{\gamma\beta} = (\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}$ and using the identity
$$(\sigma^{\mu\nu})_\alpha{}^\beta (\sigma_{\mu\nu})_\gamma{}^\delta = \epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta_\alpha^\delta \delta_\gamma^\beta$$
we have the Fierz identity
$$\psi_\alpha \chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\chi) + \frac{1}{2}(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi), \quad (1/2,0)\times(1/2,0) = (0,0)+(1,0)$$
where $\psi\chi$ is a scalar and $\psi\sigma_{\mu\nu}\chi$ is a self-dual tensor, which has the desired 3 degrees of freedom. The same kind of decomposition works for two dotted spinors.

- Another useful set of Fierz identities is
$$(\theta\psi)(\overline{\chi\eta}) = -\frac{1}{2}(\theta\sigma^\mu\overline{\eta})(\overline{\chi}\overline{\sigma}_\mu\psi), \quad (\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\overline{\theta\theta}).$$
To use these identities, it is useful to 'reorder' fields. We have
$$\theta\sigma^\mu\overline{\chi} = -\overline{\chi}\overline{\sigma}^\mu\theta, \quad \theta\sigma^\mu\overline{\sigma}^\nu\chi = \chi\sigma^\nu\overline{\sigma}^\mu\theta$$
which implies that
$$\psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi.$$
The pattern continues with alternating signs for more σ's, i.e. with everything reversed in order with σ and $\overline{\sigma}$ interchanged.

Next, we make contact with four-component Dirac spinors.

- A Dirac spinor Ψ is the direct sum of two Weyl spinors ψ and $\overline{\chi}$ of opposite chirality,
$$\Psi = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}.$$
Here the left-handed component is on top and the right-handed component is on the bottom.

- The analogue of the matrices σ^μ are the Clifford matrices
$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$
which also form a Clifford algebra. Then they similarly yield a representation of the Lorentz group, with the generators
$$\Sigma^{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}$$
which is naturally block-diagonal.

- We define the chiral matrix
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
giving the projection operators
$$P_L = \frac{1}{2}(1-\gamma^5), \quad P_R = \frac{1}{2}(1+\gamma^5).$$
We can also see this works because $\{\gamma^5, \gamma^\mu\} = 0$, so $[\gamma^5, \Sigma^{\mu\nu}] = 0$ and Lorentz transformations preserve chirality.

- We define the Dirac conjugate $\overline{\Psi}$ and the charge conjugate Ψ^C by

$$\overline{\Psi} = (\chi^\alpha, \overline{\psi}_{\dot{\alpha}}) = \Psi^\dagger \gamma^0, \quad \Psi^C = C\overline{\Psi}^T = \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

  Then charge conjugation simply exchanges χ and ψ. Majorana spinors have $\psi = \chi$ and hence are mapped to themselves under charge conjugation.

- Note that we have the gamma matrix identities

$$\Sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\gamma^5 \Sigma_{\rho\sigma}, \quad \operatorname{tr} \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4i\epsilon^{\mu\nu\rho\sigma}.$$

1.4 Supersymmetric Quantum Mechanics

In this section we'll give a preview of the following results for the case of a one-dimensional quantum field theory, i.e. quantum mechanics.

- Formally, a superalgebra over $\mathbb{C}$ is a $\mathbb{Z}_2$ graded vector space

$$A = A_0 \oplus A_1,$$

  whose two components are called the bosonic and fermionic subalgebras respectively, with a bilinear multiplication operator so that

$$a_0 a_0' \in A_0, \quad a_0 a_1 \in A_1, \quad a_1 a_1' \in A_0$$

  if a_0, a_0' are bosonic and a_1, a_1' are fermionic.

- A supersymmetry algebra over $\mathbb{R}^3$ is a superalgebra which contains the d-dimensional Poincare symmetry algebra $\mathrm{iso}(1, d-1)$ as a subalgebra of its bosonic subalgebra; below we will restrict to the case where $\mathrm{iso}(1, d-1)$ is precisely the bosonic subalgebra.

- In one dimension, the Poincare algebra has a single generator,

$$E = -i\frac{d}{dt}.$$

  We introduce $\mathcal{N}$ supersymmetry generators Q^I. In one dimension, these generators do not need an additional spinor index. It turns out that

$$[Q^I, E] = 0$$

  which indicates that all the states in an irrep of the supersymmetry algebra (a supermultiplet) are degenerate.

- The anticommutators between the SUSY generators are generally

$$\{Q^I, Q^J\} = 2E\delta^{IJ} + Z^{IJ}$$

  where the central charge Z^{IJ} is real and symmetric. Central charges are not elements of the original Lie algebra, and are taken to commute with all elements of it. They arise naturally when we allow for projective representations, as we encountered for the Galilean group in the notes on Group Theory.

One simple $\mathcal{N}=1$ supersymmetric model is a theory with D free 1D bosons $X^\mu(t)$ and D free 1D fermions $\psi^\mu(t)$, all of vanishing mass.

- The Lagrangian is
$$L = \frac{1}{2}\dot{X}_\mu \dot{X}^\mu + i\psi_\mu \dot{\psi}^\mu$$
where the fermions ψ^μ are Grassmann-valued. If the indices above are raised and lowered with the Minkowski metric, then this system represents a relativistic spinning massless particle in flat D-dimensional spacetime.

- The conjugate momenta are
$$\Pi_X^\mu = \dot{X}^\mu, \quad \Pi_\psi^\mu = i\psi^\mu.$$
In canonical quantization, we hence have
$$[X^\mu, \dot{X}^\nu] = i\eta^{\mu\nu}, \quad \{\psi^\mu, \psi^\nu\} = \eta^{\mu\nu}.$$
The latter result tells us the spin degrees of freedom match those of a Dirac spinor.

- The Lagrangian has a 1D $\mathcal{N}=1$ supersymmetry, acting on the fields as
$$\delta X = 2i\epsilon\psi, \quad \delta\psi = -\epsilon\dot{X}$$
where ϵ is a Grassmann parameter. Under this transformation the Lagrangian changes by a total derivative,
$$\delta L = i\epsilon\frac{d}{dt}\left(\psi_\mu \dot{X}^\mu\right).$$
Applying Noether's theorem, we find the conserved charge and supersymmetry generator
$$Q = \psi_\mu \dot{X}^\mu.$$

- The energy operator is
$$E = -P_0 = \frac{1}{2}\dot{X}^2.$$
The operators Q and E hence form a supersymmetry algebra, obeying the appropriate anti-commutation relations
$$\{Q, Q\} = \{\psi_\mu, \psi_\nu\}\dot{X}^\mu \dot{X}^\nu = \dot{X}^2 = 2E.$$

- We can also confirm this relation at the field level, defining $\delta = \epsilon Q$. Since we have
$$\delta^2 X = -2i\epsilon^2\dot{X}, \quad \delta^2\psi = -2i\epsilon^2\dot{\psi}$$
then we have $\{Q, Q\} = 2E$ when acting on the fields.

- Upon quantization, the state space is a tensor product of spin and spatial degrees of freedom. The operators ψ^μ serve as spin raising and lowering operators, with the total dimensionality of this space matching the degrees of freedom of a single Dirac spinor particle.

* The operators X^μ and $\dot{X}^\mu$ act on $L^2(\mathbb{R}^{D-1})$. They do not have a conventional mode expansion, because the equation of motion is $\partial^2 X^\mu = 0$, rather than something like $\partial^2\phi = 0$. The solutions are instead linear functions of time,

$$X^\mu(t) = X^\mu(0) + \dot{X}^\mu(0)t, \quad \dot{X}^\mu(t) = \dot{X}^\mu(0).$$

  Algebraically, they behave like the usual operators nonrelativistic quantum mechanics. That is, while the fields X^μ are massless, they represent a nonrelativistic particle with unit mass.

* Like other symmetries, we may find the irreducible representations of the symmetry algebra; the Hilbert space will then be built out of these "supermultiplets". Since $[Q, H] = 0$, we may restrict to states of energy E. Defining the rescaling $b = Q/\sqrt{2E}$, the $\mathcal{N} = 1$ SUSY algebra is

$$\{b, b\} = 1, \quad b = b^\dagger.$$

  The irreducible representations are all one-dimensional, so this gives us little information.

* For $\mathcal{N} = 2n$ supersymmetry, we get nontrivial information. Define the complex supercharges

$$\mathcal{Q}^i = Q^i + iQ^{n+i}, \quad \overline{\mathcal{Q}}^i = Q^i - iQ^{n+i}, \quad i = 1, \ldots, n.$$

  The supersymmetry algebra then takes the form

$$\{\mathcal{Q}^i, \overline{\mathcal{Q}}^j\} = 4E\delta^{ij}, \quad \{\mathcal{Q}^i, \mathcal{Q}^j\} = 2i\mathcal{Z}^{ij}, \quad \{\overline{\mathcal{Q}}^i, \overline{\mathcal{Q}}^j\} = -2i\mathcal{Z}^{ij}, \quad \mathcal{Z}^{ij} = Z^{i,n+j}$$

  In the case of vanishing central charges, if we define

$$a_i = \frac{\mathcal{Q}_i}{2\sqrt{E}}, \quad a_i^\dagger = \frac{\overline{\mathcal{Q}}_i}{2\sqrt{E}}$$

  then we have n independent fermionic QHOs, and hence a supermultiplet with 2^n states.

We will see much of this again in more detail for $d = 4$, though there will be additional complications, such as spinor indices.

1.5 Spinors in Various Dimensions

Though we will mostly focus on four dimensions, it is useful to set conventions for spinors in arbitrary dimension. We've already covered spinor representations of $\mathfrak{so}(n)$ in the notes on Group Theory, but here will focus on spinor representations in Minkowski space, which differ in several respects.

* We focus on $\mathfrak{so}(1, d-1)$, where the signature is mostly positive. We take the generators $M_{\mu\nu} = -M_{\nu\nu}$ to satisfy the algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}).$$

* A Clifford algebra is a set of d matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

  The Dirac spinor representation of $\mathfrak{so}(1, d-1)$ is the one with representation matrices

$$\mathbf{M}_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

  where the γ_μ have the minimum possible dimension while still representing the Clifford algebra faithfully. Note that various factors of i may differ depending on the source.

- To see the dimension of the Dirac spinor representation, let $d = 2(k+1)$ and define
$$\gamma^{0\pm} = \frac{1}{2}(\pm\gamma^0 + \gamma^1), \quad \gamma^{a\pm} = \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}), \quad a = 1, \dots k.$$
This defines a set of $n+1$ independent fermionic QHOs,
$$\{\gamma^{a+}, \gamma^{b-}\} = \delta^{ab}, \quad \{\gamma^{a+}, \gamma^{b+}\} = \{\gamma^{a-}, \gamma^{b-}\} = 0.$$
Assuming the Clifford algebra is represented faithfully and irreducibly, this gives 2^{k+1} states.

- Given a Clifford algebra in $d = 2n$ dimensions, we can automatically construct a Clifford algebra in $d = 2n+1$ dimensions by adding
$$\gamma^{2n+1} = (-i)^{n+1}\gamma^0\gamma^1 \dots \gamma^{2n-1}.$$
Note that this means the index μ in γ^μ takes values $\{0, 1, \dots, 2n-1, 2n+1\}$, which is unfortunately conventional.

- Given a Clifford algebra γ^μ in $d = 2n$ dimensions, there are various ways to construct a Clifford algebra Γ^μ in $d = 2n+2$ dimensions by tensor product. One way to do this is
$$\Gamma^\mu = \gamma^\mu \otimes \sigma^1, \quad \Gamma^{2n+1} = \gamma^{2n+1} \otimes \sigma^1, \quad \Gamma^{2n+2} = 1 \otimes \sigma^2.$$

- When d is even, the Dirac spinor representation is reducible. Conceptually, this is because the generators $\mathbf{M}_{\mu\nu}$ are all built from an even number of γ matrices, so they preserve the parity of the number of fermionic QHO excitations. Concretely, we can extract the irreps, called Weyl spinors, using the projection operators $(1 \pm \gamma^{2n+1})/2$. In all dimensions, we call these two Weyl spinors left-chiral and right-chiral respectively; they correspond to the two special roots at the end of the Dynkin diagram for $\mathfrak{so}(2n)$. Going up to $d = 2n+1$ combines these irreps into one.

- So far, all representations have been complex, in the mathematician's sense. However, the Dirac spinor is not complex, in the physicist's sense; it is either real or pseudoreal. In the case it is real (in the physicist's sense), there is a chance we can extract a real representation (in the mathematician's sense) from it. Such a spinor is called a Majorana spinor, and it turns out to be possible if $d \equiv 0, 1, 2, 3, 4 \pmod 8$.

- The Weyl spinors may be self-conjugate, or conjugate to each other, depending on the dimension. In the case where they are self-conjugate, there is a chance we can extract a real representation from them, yielding a Majorana–Weyl spinor. This turns out to be possible if $d \equiv 2 \pmod 8$.

- The facts above are summarized in the below table.

d	$\dim\gamma$	Majorana	Weyl	Majorana–Weyl	min. dim.
2	2	yes	self	yes	1
3	2	yes			2
4	4	yes	complex		4
5	4				8
6	8		self		8
7	8				16
8	16	yes	complex		16
9	16	yes			16
10	32	yes	self	yes	16
11	32	yes			32
12	64	yes	complex		64

The columns are the total spacetime dimension, the dimension of the (complex) gamma matrices, the presence of Majorana, Weyl, and Majorana–Weyl spinors, and the real dimension of the smallest possible representation.

Example. Spinors in low dimensions. For $d = 2$, we may choose

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives a two-dimensional Dirac spinor which decomposes into one-dimensional "left-moving" and "right-moving" Weyl spinors. To move to $d = 3$ we may add

$$\gamma^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which also may be used to project out the left-movers and right-movers. Of course, the cases $d = 4$ and hence $d = 5$ are familiar, though we won't get the usual chiral representation if we use our inductive scheme above.

2 SUSY Algebra and Representations

2.1 The SUSY Algebra

Next, we deduce the SUSY algebra.

- The SUSY algebra is a graded Lie algebra. The operators in such an algebra obey

$$[O_a, O_b]_\pm \equiv O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = iC^e{}_{ab} O_e$$

where the gradings η_a take the form

$$\eta_a = \begin{cases} 0 & O_a \text{ bosonic} \\ 1 & O_a \text{ fermionic.} \end{cases}$$

Below we won't use the $[\cdot,\cdot]_\pm$ notation, but instead will make (anti)commutators explicit.

- The 'super-Jacobi identity' is

$$(-1)^{\eta_a\eta_c}[O_a,[O_b,O_c]_\pm]_\pm + (-1)^{\eta_b\eta_a}[O_b,[O_c,O_a]_\pm]_\pm + (-1)^{\eta_c\eta_b}[O_c,[O_a,O_b]_\pm]_\pm = 0.$$

Note that the signs do nothing unless exactly two of the operators are fermionic.

- For the SUSY algebra, the generators are the Poincare generators P^μ and $M^{\mu\nu}$ and the spinor generators Q^A_α and $\overline{Q}^A_{\dot\alpha} = (Q^A_\alpha)^\dagger$ where $A = 1, \ldots, \mathcal{N}$. For $\mathcal{N} = 1$, we have simple SUSY, while for $\mathcal{N} > 1$ we have extended SUSY. Here we focus on simple SUSY, which is the most phenomenologically relevant.

- The Poincare algebra still holds, so by the grading we must find

$$[Q_\alpha, M^{\mu\nu}], \quad [Q_\alpha, P^\mu], \quad \{Q_\alpha, Q_\beta\}, \quad \{Q_\alpha, \overline{Q}_{\dot\beta}\}.$$

We now consider these four in turn.

- The logic for the first is like that for $[P^\sigma, M^{\mu\nu}]$. Since Q_α is a spinor, it transforms as

$$Q'_\alpha = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \approx \left(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta.$$

On the other hand, for a representation of the super-Poincare algebra on a Hilbert space,

$$Q'_\alpha = U^\dagger Q_\alpha U \approx \left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q_\alpha \left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, Q_\alpha].$$

Comparing the two, we conclude

$$\boxed{[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta}$$

which is just the statement Q_α is a spinor. By similar reasoning,

$$[\overline{Q}^{\dot\alpha}, M^{\mu\nu}] = (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \overline{Q}^{\dot\beta}.$$

More generally, for a SUSY generator transforming in an arbitrary spinor representation, $\sigma^{\mu\nu}$ would be replaced with the representation matrices $\mathbf{M}^{\mu\nu}$.

- The spinors are translationally invariant, so on intuitive grounds

$$\boxed{[Q_\alpha, P^\mu] = [\overline{Q}^{\dot\alpha}, P^\mu] = 0.}$$

To derive this more formally, note that we must have

$$[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot\alpha}\overline{Q}^{\dot\alpha}$$

by index structure and linearity on the right-hand side. The adjoint of this equation is

$$[\overline{Q}^{\dot\alpha}, P^\mu] = c^*(\overline{\sigma}^\mu)^{\dot\alpha\beta}Q_\beta$$

and the Jacobi identity for P^μ, P^ν, and Q_α reduces to

$$|c^2|(\sigma^\nu\overline{\sigma}^\mu - \sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta Q_\beta = 0$$

which can only hold in general if $c = 0$.

- Next, consider $\{Q_\alpha, Q_\beta\}$. This transforms in the Lorentz representation $(1/2, 0) \times (1/2, 0) = (1, 0) + (0, 0)$, but the $(0, 0)$ piece vanishes because the Q_α are anticommuting. Then

$$\{Q_\alpha, Q^\beta\} = k(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}$$

for an arbitrary constant k, where σ carries the appropriate $SL(2, \mathbb{C})$ indices and $M_{\mu\nu}$ is the only thing that can absorb its Lorentz indices. By the Jacobi identity, the left-hand side commutes with P^μ but the right-hand side does not unless $k = 0$, so we must have

$$\boxed{\{Q_\alpha, Q_\beta\} = 0.}$$

- Finally, for $\{Q_\alpha, \overline{Q}_{\dot\beta}\}$ we have $(1/2, 0) \times (0, 1/2) = (1/2, 1/2)$, so

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = t(\sigma^\mu)_{\alpha\dot\beta}P_\mu.$$

There is no way to fix t. If we set $t = 0$, the algebra is trivial since the spinor and Poincare parts are completely independent. Then by convention we set $t = 2$ for

$$\boxed{\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu.}$$

Remarkably, this means that $Q\overline{Q}$ is a translation! That is, if we start with a bosonic/fermionic state and act with $Q\overline{Q}$, we get back a translated bosonic/fermionic state.

- Finally, let T_i generate an internal symmetry. Then usually we must have $[Q_\alpha, T_i] = 0$. The exception is the $U(1)$ automorphism of the supersymmetry algebra called R symmetry,

$$Q_\alpha \to e^{i\lambda}Q_\alpha, \quad \overline{Q}_{\dot\alpha} \to e^{-i\lambda}\overline{Q}_{\dot\alpha}.$$

If R generates this symmetry then

$$[Q_\alpha, R] = Q_\alpha, \quad [\overline{Q}_{\dot\alpha}, R] = -\overline{Q}_{\dot\alpha}.$$

A specific SUSY theory may or may not have this R-symmetry, depending on the Lagrangian; furthermore R-symmetry may be anomalous. As we'll see later, postulating an R-symmetry is useful for constraining the MSSM.

2.2 $\mathcal{N} = 1$ SUSY Representations

Next, we turn to representations of the SUSY algebra. We begin by reviewing Wigner's classification.

- Recall that for $\mathfrak{su}(2)$, we have a Casimir operator J^2 which labels the irreps; states in the irreps are labeled by J_z.
- In the Poincare algebra, the Pauli–Lubanski vector is a "generalized spin",
$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$$
and it obeys the commutation relations
$$[W_\mu, P_\nu] = 0, \quad [W_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho), \quad [W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.$$
The first two simply say that W_μ is a translationally-invariant vector, while the third indicates it does not form a closed algebra. In verifying these results it is useful to use the identity
$$\epsilon^{a_1\ldots a_p c_{p+1}\ldots c_n}\epsilon_{b_1\ldots b_p c_{p+1}\ldots c_n} = -p!(n-p)!\delta^{a_1}_{[b_1}\ldots\delta^{a_p}_{b_p]}.$$
- As a result, the Poincare Casimirs are
$$C_1 = P^\mu P_\mu, \quad C_2 = W^\mu W_\mu.$$
The eigenvalue of C_1 is written as m^2, where m is the mass of the particle.
- Next, we find the irreps using the little group. To use this method, we fix a reference momentum p^μ and look at the subalgebra that preserves the momentum; in the Poincare group the only such operators are the W_μ, which take the form
$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}p^\nu M^{\rho\sigma}.$$
- In the massive case, we have
$$p^\mu = (m, 0, 0, 0), \quad W_0 = 0, \quad W_i = -mJ_i.$$
Then the little group is $SO(3)$, and C_2 indexes the spin.
- In the massless case, we have
$$p^\mu = (E, 0, 0, E), \quad W_\mu = E(J_3, -J_1 + K_2, -J_2 - K_1, -J_3)$$
which have the commutation relations
$$[W_1, W_2] = 0, \quad [W_3, W_1] = -iEW_2, \quad [W_3, W_2] = iEW_1$$
of the Euclidean group in two dimensions, E_2, which has infinite-dimensional irreps which are not seen in nature. Concentrating on the finite-dimensional representations, the translations must act trivially, leaving $SO(2)$. The irreps are labeled by the helicity λ where $W^\mu = \lambda P^\mu$, and projective representations allow half-integer λ.

Next, we extend these results to the SUSY algebra.

- Since the SUSY generators commute with P^μ, C_1 remains a Casimir operator, so all particles in a SUSY multiplet have the same mass. Now, for $\mathcal{N}=1$ SUSY, we have

$$[W_\mu, Q_\alpha] = -iP_\nu(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta$$

which means that C_2 is no longer a Casimir operator. This is as expected, as we can have particles of different spin inside a SUSY multiplet.

- Instead, we define the operators

$$B_\mu = W_\mu - \frac{1}{4}\overline{Q}_{\dot\alpha}(\overline{\sigma}_\mu)^{\dot\alpha\beta}Q_\beta, \quad C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu$$

which yields a Casimir operator, called the superspin,

$$\tilde{C}_2 = C_{\mu\nu}C^{\mu\nu}.$$

- Next, we claim that in any SUSY multiplet the number n_B of bosonic states equals the number n_F of fermionic states. Consider the fermion number operator $(-)^F$, defined by

$$(-)^F|B\rangle = |B\rangle, \quad (-)^F|F\rangle = -|F\rangle.$$

This operator anticommutes with Q_α as, e.g. we have

$$(-)^F Q_\alpha|F\rangle = (-)^F|B\rangle = |B\rangle = Q_\alpha|F\rangle = -Q_\alpha(-)^F|F\rangle.$$

Next we consider the trace

$$\mathrm{tr}(-)^F\{Q_\alpha, \overline{Q}_{\dot\beta}\} = \mathrm{tr}(-)^F Q_\alpha\overline{Q}_{\dot\beta} + \mathrm{tr}(-)^F\overline{Q}_{\dot\beta}Q_\alpha = 0$$

by the anticommutation relation and the cyclic property of the trace. On the other hand,

$$\mathrm{tr}(-)^F\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\,\mathrm{tr}(-)^F(\sigma^\mu)_{\alpha\dot\beta}P_\mu = 2(\sigma^\mu)_{\alpha\dot\beta}p_\mu\,\mathrm{tr}(-)^F$$

where in the last step we restricted to states with momentum p^μ. This can only hold if

$$0 = \mathrm{tr}(-)^F = \sum_B\langle B|(-)^F|B\rangle + \sum_F\langle F|(-)^F|F\rangle = \sum_B\langle B|B\rangle - \sum_F\langle F|F\rangle = n_B - n_F.$$

- There is an exception to this reasoning. If the supersymmetry is not broken, then the vacuum states have $p_\mu = 0$, as we'll see below. Then the trace $\mathrm{tr}(-)^F$ evaluated over the entire Hilbert space may be nonzero; it is called the Witten index. It is important because it is preserved under certain deformations of the theory.

We now construct the massless SUSY multiplets. These are the most relevant phenomenologically as almost all particles in the SM are 'really' massless, only acquiring mass from the Higgs.

- We take the reference momentum to be $p_\mu = (E, 0, 0, E)$ and consider states in this irrep with the reference momentum $|p^\mu, \lambda\rangle$, where λ stands for all quantum numbers. The Casimirs $C_1 = P^\mu P_\mu$ and $\tilde{C}_2 = C_{\mu\nu}C^{\mu\nu}$ are both zero. We already know that the Poincare generators don't give any new states, so we focus on the spinors.

- Note that among the states $|p^\mu, \lambda\rangle$,

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu = 2E(\sigma^0 + \sigma^3) = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta}.$$

Therefore, we have

$$\langle p^\mu, \lambda | \{Q_2, \overline{Q}_{\dot 2}\} | p^\mu, \lambda\rangle = 0$$

which can only hold if $Q_2 |p^\mu, \lambda\rangle = 0$.

- Meanwhile, the Q_1 satisfy $\{Q_1, \overline{Q}_{\dot 1}\} = 4E$, so defining

$$a = \frac{Q_1}{2\sqrt{E}}, \quad a^\dagger = \frac{\overline{Q}_{\dot 1}}{2\sqrt{E}}, \quad \{a, a^\dagger\} = 1, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0$$

which are the commutation relations for a fermionic harmonic oscillator.

- Using the SUSY algebra, we may show that

$$[Q_\alpha, J_i] = \frac{1}{2} (\sigma_i)^\beta{}_\alpha Q_\beta, \quad [\overline{Q}^{\dot\alpha}, J_i] = \frac{1}{2} (\sigma_i)^{\dot\alpha}{}_{\dot\beta} \overline{Q}^{\dot\beta}$$

where the σ_i with indices in these positions are just ordinary Pauli matrices. Then

$$[a^\dagger, J^3] = \frac{1}{2} (\sigma^3)_{22} a^\dagger = -\frac{1}{2} a^\dagger.$$

Here, care must be taken with the index positions, noting that $\overline{Q}_{\dot 1} = -\overline{Q}^{\dot 2}$ and $\overline{Q}_{\dot 2} = \overline{Q}^{\dot 1}$.

- Thus, we see that $a^\dagger$ raises the J^3 eigenvalue by $1/2$. Since the particle is moving in the $-z$ direction, it lowers the helicity λ by $1/2$.

- We let $|\Omega\rangle = |p^\mu, \lambda\rangle$ be the state of highest helicity. Then we get just one other state,

$$a^\dagger |\Omega\rangle = |p^\mu, \lambda - 1/2\rangle.$$

As before, CPT flips λ, so we get irreps where the states have helicities $\{\pm\lambda, \pm(\lambda - 1/2)\}$.

- We have chiral multiplets with $\lambda = 0, 1/2$, examples being

$$(\text{squark}, \text{quark}), \quad (\text{slepton}, \text{lepton}), \quad (\text{Higgs}, \text{Higgsino})$$

along with vector/gauge multiplets with $\lambda = 1/2, 1$, examples being

$$(\text{photino}, \text{photon}), \quad (\text{gluino}, \text{gluon}), \quad (\text{Wino}, W), \quad (\text{Zino}, Z).$$

In general, the $\lambda = 1/2$ components are called gauginos. The SM matter fields can't be gauginos, because both particles in a vector multiplets transform the same way under $SU(3)_c \times SU(2)_L \times U(1)_Y$, and vector particles must be created by gauge fields, which transform in the adjoint. Finally we have

$$\lambda = (3/2, 2): \quad (\text{gravitino}, \text{graviton}).$$

By the same argument as in Wigner's classification, we need only consider half-integer λ.

Next, we consider massive supermultiplets, which are somewhat more complicated.

- In the massive case we have $p^\mu = (m, 0, 0, 0)$ with Casimirs

$$C_1 = P^\mu P_\mu = m^2, \quad \tilde{C}_2 = 2m^4 Y^i Y_i$$

where Y_i is the superspin,

$$Y_i = J_i - \frac{1}{4m}\overline{Q}\overline{\sigma}_i Q = \frac{B_i}{m}, \quad [Y_i, Y_j] = i\epsilon_{ijk} Y_k.$$

The eigenvalues of $Y^2 = Y^i Y_i$ are thus $y(y+1)$, and we label irreps by m and y.

- Again restricting to states in an irrep with momentum p^μ, we have

$$\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu = 2m(\sigma^0)_{\alpha\dot{\beta}} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}}.$$

Therefore, we have two sets of fermionic ladder operators,

$$a_{1,2} = \frac{Q_{1,2}}{\sqrt{2m}}, \quad a_{1,2}^\dagger = \frac{\overline{Q}_{\dot{1},\dot{2}}}{\sqrt{2m}}, \quad \{a_p, a_q^\dagger\} = \delta_{pq}$$

which means that, starting from a vacuum state, we can build 4 states instead of 2.

- We let $|\Omega\rangle$ be a 'vacuum' state, annihilated by $a_{1,2}$. Then

$$Y_i|\Omega\rangle = J_i|\Omega\rangle - \frac{1}{4m}\overline{Q}\overline{\sigma}_i\sqrt{2m}a|\Omega\rangle = J_i|\Omega\rangle$$

so for a vacuum state the spin j and superspin y coincide,

$$|\Omega\rangle = |m, j = y, p^\mu, j_3\rangle.$$

We can get all j_3 values by Lorentz transformations, so there are $2y+1$ vacuum states.

- Since the SUSY generators carry spin 1/2, they act on spin $j = y$ states to yield states of spin $j = y \pm 1/2$. Using the same relations as above, we find the SUSY generators change j_3 by

$$[a_1^\dagger, J^3] = -\frac{1}{2}a_1^\dagger, \quad [a_2^\dagger, J^3] = -\frac{1}{2}a_2^\dagger$$

so that $a_1^\dagger$ raises J_3 as in the massless case and $a_2^\dagger$ lowers it. Then it can be shown that

$$[J^2, \overline{Q}^{\dot{\alpha}}] = \frac{3}{4}\overline{Q}^{\dot{\alpha}} - (\sigma_i)^{\dot{\alpha}}_{\dot{\beta}}\overline{Q}^{\dot{\beta}} J_i, \quad [J_3, a_1^\dagger a_2^\dagger] = [J^2, a_1^\dagger a_2^\dagger] = 0.$$

The last identity states that acting with both $a_1^\dagger$ and $a_2^\dagger$ does not change J^2.

- Therefore for $y > 0$, we have

$$a_1^\dagger|j = y, j_3\rangle = k_1|j = y + 1/2, j_3 + 1/2\rangle + k_2|j = y - 1/2, j_3 + 1/2\rangle$$

and

$$a_2^\dagger|j = y, j_3\rangle = k_3|j = y + 1/2, j_3 - 1/2\rangle + k_4|j = y - 1/2, j_3 - 1/2\rangle$$

where we suppress m and p^μ and the k_i are Clebsch–Gordan coefficients.

- Finally, we have spin j states of the form $|\Omega'\rangle = a_2^\dagger a_1^\dagger |\Omega\rangle$. These are not proportional to $|\Omega\rangle$, since the a_i annihilate $|\Omega\rangle$ but not $|\Omega'\rangle$. Thus we have

$$\text{two particles of spin } j = y, \quad \text{one particle each of spin } j = y \pm 1/2$$

  which gives $n_F = n_B = 2(2y+1)$ as expected. The total number of physical states in the multiplet is $4(2y+1)$, i.e. 4 times the $2y+1$ vacuum states.

- The case $y = 0$ is slightly different. In this case we have $y \otimes 1/2 = 1/2$, i.e. we have one particle of spin 1/2 and two particles of spin 0. Again we have $4(2y+1) = 4$ states, but only three $SO(3)$ irreps.

- Finally, we consider parity transformations. Parity exchanges the Lorentz representations $(1/2, 0)$ and $(0, 1/2)$, and since $\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu$ we must have

$$\widehat{P} Q_a \widehat{P}^{-1} = \eta_P (\sigma^0)_{\alpha\dot\beta} \overline{Q}^{\dot\beta}, \quad \widehat{P} \overline{Q}^{\dot\alpha} \widehat{P}^{-1} = \eta_P^* (\overline{\sigma}^0)^{\dot\alpha\beta} Q_\beta$$

  where η_P is a phase factor, and we are using the identity matrices σ^0 and $\overline{\sigma}^0$ just to make the indices match up. As a result,

$$\widehat{P} P^\mu \widehat{P}^{-1} = (P^0, -\mathbf{P}), \quad \widehat{P}^2 Q \widehat{P}^{-2} = -Q.$$

- Heuristically, parity exchanges Q and $\overline{Q}$. Note that the a_i annihilate $|\Omega\rangle$ and the $a_i^\dagger$ annihilate $|\Omega'\rangle$. Then parity exchanges the highest and lowest states. The states with definite parity are

$$|\pm\rangle = |\Omega\rangle \pm |\Omega'\rangle, \quad P|\pm\rangle = \pm|\pm\rangle$$

  where $|+\rangle$ is scalar and $|-\rangle$ is pseudoscalar.

Note. A common confusion with supersymmetry is the following: "Supersymmetry is a symmetry that swaps fermions and bosons. But it can't be true, because a state with two identical bosons can't map to anything, by the Pauli exclusion principle." The reason this is misguided is that the intuition of symmetry transformations leaving a system the same only applies for symmetry groups. We never work with a "SUSY group", only with the SUSY algebra. Applying an operator in a symmetry algebra (such as Q, but also $L_\pm$) represents an infinitesimal *change* in state, rather than a new state related by symmetry. The reasoning above really tells us why we can't exponentiate the SUSY algebra into a SUSY group: the resulting transformations wouldn't be invertible.

2.3 Extended SUSY

Next, we turn to the case of extended SUSY, $\mathcal{N} > 1$.

- The SUSY algebra remains the same, but the anticommutation relations between SUSY generators are modified. Before, we only showed that $\{Q_\alpha^A, Q_\beta^B\}$ could not depend on P_μ or $M_{\mu\nu}$. (In fact, in general dimension, the anticommutator can also depend on P_μ.) However, for $\mathcal{N} > 1$ it is consistent to include a central charge,

$$\boxed{\{Q_\alpha^A, \overline{Q}_{\dot\beta B}\} = 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \delta_B^A, \quad \{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB}.}$$

  The central charges Z^{AB} are bosonic and antisymmetric; they commute with all of the generators and with each other. Thus they form an abelian invariant subalgebra of internal symmetries.

- Specifically, if G is the set of internal symmetries, define the R-symmetry group $H \subset G$ to be the set that do not commute with the supersymmetry generators, i.e. the ones that change the Q^A_α nontrivially. By the Coleman–Mandula theorem, they do not affect the Poincare generators.
- In the case $Z^{AB} = 0$, the R-symmetry group is $U(N)$, e.g.
$$Q^A_\alpha \to U^A{}_B Q^B_\alpha, \quad \overline{Q}^A_{\dot\alpha} \to (U^*)^A{}_B \overline{Q}^B_{\dot\alpha}$$
generalizing the $U(1)$ symmetry found earlier. That is, the Q^A transform in the fundamental and the $\overline{Q}^A$ in the antifundamental.
- The full R-symmetry need not be realized, depending on the theory. For example, the maximal R-symmetry in $\mathcal{N} = 4$ is $U(4)$, but the actually realized symmetry in $\mathcal{N} = 4$ SYM is $SU(4)$.

Next, we proceed to the massless irreps.

- Again taking $p_\mu = (E, 0, 0, E)$, we have
$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \delta^A_B$$
which again implies that $Q^A_2 |p^\mu, \lambda\rangle = 0$, and hence $Z^{AB}|p^\mu, \lambda\rangle = 0$.
- We now define $\mathcal{N}$ sets of fermionic creation and annihilation operators,
$$a^{A\dagger} = \frac{Q^A_1}{2\sqrt{E}}, \quad a^A = \frac{\overline{Q}^A_{\dot 1}}{2\sqrt{E}}, \quad \{a^A, a^\dagger_B\} = \delta^A_B$$
where we flip the convention here for convenience. We start with the vacuum $|\Omega\rangle$ with helicity λ_0. Then the $\mathcal{N}$ states
$$a^{A\dagger}|\Omega\rangle$$
all have helicity $\lambda_0 + 1/2$. More generally we may act with k creation operators, giving $\binom{\mathcal{N}}{k}$ states with helicity $\lambda_0 + k/2$, for a total of $2^{\mathcal{N}}$ states.
- For example, for $\mathcal{N} = 2$ and $\lambda_0 = 0$ we have the vector multiplet
$$\lambda = 0, \quad 2 \times \lambda = 1/2, \quad \lambda = 1.$$
We could also keep track of the R-symmetry; in this case the $\lambda = 1/2$ transform in the 2 of $U(2)$ while the others transform in the 1.
- Upon restriction to $\mathcal{N} = 1$, the $\mathcal{N} = 2$ vector multiplet decomposes into an $\mathcal{N} = 1$ vector and chiral multiplet. Note that these multiplets depend on which of the two supersymmetries we restrict to! In a CPT symmetric theory, the vector multiplet must be accompanied with its CPT conjugate.
- Next, for $\mathcal{N} = 2$ and $\lambda_0 = -1/2$, we have the hyper multiplet
$$\lambda = -1/2, \quad 2 \times \lambda = 0, \quad \lambda = 1/2$$
which decomposes into two $\mathcal{N} = 1$ chiral multiplets.

- It's tempting to conclude this multiplet could be its own CPT conjugate (as long as, e.g. there are no additional complex internal quantum numbers). However, it turns out that the $\lambda = 0$ sector must be a real representation of the R-symmetry group, while in this case it is pseudoreal. (Central charges make little difference here, as they only break the R-symmetry group from $U(2)$ to $SU(2)$.) Hence the hyper multiplet contains two copies of the helicities above, while only one copy is a "half hyper multiplet".

- Next, for $\mathcal{N} = 4$ and $\lambda_0 = -1$, we have the vector multiplet

$$\lambda = -1, \quad 4 \times \lambda = -1/2, \quad 6 \times \lambda = 0, \quad 4 \times \lambda = 1/2, \quad \lambda = 1.$$

  This is the only $\mathcal{N} = 4$ multiplet where $|\lambda| \leq 1$. Restricting to $\mathcal{N} = 2$, we get two $\mathcal{N} = 2$ vector multiplets and hypermultiplets. Restricting to $\mathcal{N} = 1$, we get two $\mathcal{N} = 1$ vector multiplets and six $\mathcal{N} = 1$ chiral multiplets. Since the whole $\mathcal{N} = 4$ multiplet is symmetric under $\lambda \to -\lambda$, and the 6 of $U(4)$ is real, it could be its own CPT conjugate, in which case the submultiplets pair up under CPT.

- Finally, for $\mathcal{N} = 8$ and $\lambda_0 = -2$, we have the 'maximum multiplet' or 'gravity multiplet'

$$\lambda = \pm 2, \quad 8 \times \lambda = \pm 3/2, \quad 28 \times \lambda = \pm 1, \quad 56 \times \lambda = \pm 1/2, \quad 70 \times \lambda = 0.$$

  Since the 28 of $U(8)$ is real, this multiplet again could be its own CPT conjugate. The general rule is that this holds when the maximum helicity is $\mathcal{N}/4$, so we've exhausted the realistic self-CPT conjugate possibilities above.

We now comment on the physical properties of the particles in these multiplets.

- Note that renormalizable field theories must have $|\lambda| \leq 1$, since otherwise the propagator does not fall off fast enough, so we require $\mathcal{N} \leq 4$ for renormalizability. This doesn't mean such theories are physically irrelevant, as gravity isn't renormalizable either.

- Generally, a massless particle with $|\lambda| \geq 1$ must couple to a conserved current, i.e. a conserved vector for $\lambda = \pm 1$ (as in electromagnetism) and a conserved tensor for $\lambda = \pm 2$ (as in gravity). This is necessary to remove the growth in the propagators, which would otherwise violate perturbative unitarity.

- There aren't conserved tensors of higher rank by the Coleman–Mandula theorem and its generalizations. Thus, $|\lambda| > 2$ is forbidden, and we can only have one particle with $\lambda = 2$ because all such particles must act like gravitons. Then $\mathcal{N} = 8$ is the maximum realistic number of supersymmetries.

- Theories with massless particles of helicity $|\lambda| > 2$ are called higher-spin theories and are rather exotic. To be realized in quantum field theory, they must either be free, or contain an infinite tower of particles of essentially every spin.

- In light of the above, $\mathcal{N} = 4$ is the 'nicest' for gauge theory and $\mathcal{N} = 8$ is the 'nicest' for gravity, explaining why $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SUGRA are so well studied.

- However, $\mathcal{N} > 1$ supersymmetry is 'non-chiral', in contradiction with the Standard Model. First, note that with the sole exception of the $\mathcal{N} = 2$ hypermultiplet, all such multiplets contain $\lambda = \pm 1$ particles.

- It can be argued generally that these particles must be gauge bosons; as a result, they must transform in the adjoint representation, which is in general a real representation. (For matrix Lie groups, this is easy to see, since it is essentially the fundamental times the antifundamental.) Since internal symmetries commute with SUSY transformations, the $\lambda = \pm 1/2$ particles must also transform in the adjoint.

- If the multiplet contains both $\lambda = \pm 1/2$, then these particles transform in the same representation, so we cannot get a chiral theory; this accounts for the $\mathcal{N} = 2$ hypermultiplet.

- On the other hand, if the multiplet contains only, say, $\lambda = 1/2$, then by CPT there must be another multiplet with $\lambda = -1/2$ which transforms in the conjugate representation. Since the adjoint is real, the $\lambda = -1/2$ particle also transforms in the adjoint, and we again don't have a chiral theory.

- Helicity $\lambda = 3/2$ is also somewhat exotic. It turns out that it must couple to the supersymmetry current (i.e. the current whose charge corresponds to the SUSY generators) and be associated with a field with gauge symmetry, where the gauge symmetry is *local* supersymmetry. This necessitates local super-Poincare gauge symmetry, i.e. supergravity, so a theory with a $\lambda = 3/2$ particle (called a gravitino) must also have a $\lambda = 2$ particle.

Finally, we consider the massive multiplets.

- We consider $p_\mu = (m, 0, 0, 0)$ which gives

$$\{Q_\alpha^A, \overline{Q}_{\dot{\beta}B}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_B^A.$$

  We first consider the case where all the central charges vanish. Then we have $2\mathcal{N}$ pairs of fermionic creation and annihilation operators,

$$a_\alpha^A = \frac{Q_\alpha^A}{\sqrt{2m}}, \quad a_{\dot{\alpha}}^{A\dagger} = \frac{\overline{Q}_{\dot{\alpha}}^A}{\sqrt{2m}}$$

  which yields a much larger multiplet containing $2^{2\mathcal{N}}$ states for each vacuum state, for a total of $(2y+1)2^{2\mathcal{N}}$. As before, each of the raising operators changes the spin by $1/2$, while the combination $a_{\dot{1}}^{A\dagger} a_{\dot{2}}^{A\dagger}$ does not change the spin.

- For example, for $\mathcal{N} = 2$ with a spin 0 vacuum, at each level of raising we have

$$1 \times \text{spin } 0 \to 4 \times \text{spin } 1/2 \to 3 \times \text{spin } 0, 3 \times \text{spin } 1 \to 4 \times \text{spin } 1/2 \to 1 \times \text{spin } 0$$

  which gives 16 total states, with 5 spin 0 particles, 4 spin 1/2 particles, and 1 spin 1 particle. There are 8 fermionic states and 8 bosonic states, as expected. However, note that the number of fermionic Poincare irreps does not match the number of bosonic Poincare irreps.

- In the case $Z^{AB} \neq 0$, the size of the multiplets depends on the central charges. It is simplest to begin with $\mathcal{N} = 2$, where the central charge has one degree of freedom. Then we may take

$$\{Q_\alpha^A, \overline{Q}_{\dot{\beta}B}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta_B^A, \quad \{Q_1^A, Q_2^B\} = 2Z\epsilon^{AB}, \quad \{\overline{Q}_{\dot{1}}^A, \overline{Q}_{\dot{2}}^B\} = 2Z\epsilon^{AB}$$

  with all other anticommutators vanishing.

- We need to perform a change of basis to yield fermionic QHOs. To do this, we take

$$a_1 = \frac{1}{\sqrt{2}}(Q_1^1 + \alpha_1 \overline{Q}_2^2), \quad a_2 = \frac{1}{\sqrt{2}}(Q_2^1 - \alpha_2 \overline{Q}_1^2), \quad b_1 = \frac{1}{\sqrt{2}}(Q_1^1 - \alpha_1 \overline{Q}_2^2), \quad b_2 = \frac{1}{\sqrt{2}}(Q_2^1 + \alpha_2 \overline{Q}_1^2)$$

  along with the conjugates, using $(Q_\alpha^A)^\dagger = \overline{Q}_{\dot{\alpha}}^A$. These are constructed so that all anticommutators vanish except for

$$\{a_i, a_i^\dagger\} = 2m + \alpha_i \overline{Z} + \overline{\alpha}_i Z, \quad \{b_i, b_i^\dagger\} = 2m - \alpha_i \overline{Z} - \overline{\alpha}_i Z, \quad \{a_i, b_i^\dagger\} = \alpha_i \overline{Z} - \overline{\alpha}_i Z$$

  for $i = 1, 2$.

- In order to get an independent set of fermionic QHOs, we may choose

$$\alpha_1 = \alpha_2 = e^{i \arg Z}$$

  upon which the nonzero anticommutators become

$$\{a_i, a_i^\dagger\} = 2m + 2|Z|, \quad \{b_i, b_i^\dagger\} = 2m - 2|Z|.$$

  Since the left-hand sides are positive definite, we have the BPS bound

$$|Z| < m.$$

- If the BPS bound is not saturated, then upon a rescaling, we have four independent fermionic QHOs and get 16 degrees of freedom as before. However, $Z = m$, then b_i must be realized trivially on the multiplet, so we only get 4 degrees of freedom.

- More generally, for even $\mathcal{N}$ we can diagonalize Z^{AB} to 2×2 blocks of the above form, with values Z_1 through $Z_{\mathcal{N}/2}$. There is a BPS bound for each individual block, $2m \geq Z_i$.

- If none of these bounds are saturated, we get a "long multiplet" of $2^{2\mathcal{N}}$ states. If k of them are, we get a "short multiplet" of $2^{2(\mathcal{N}-k)}$ states, by the same logic as the $\mathcal{N} = 2$ case. If all of them are, we get an "ultra-short multiplet" of $2^{\mathcal{N}}$ states.

- Historically, BPS bounds and states were first found for soliton/monopole solutions of the Yang–Mills equations. The BPS states are stable because they are the lightest charged particles.

- Extremal black holes are also BPS states in extended supergravity theories. They are stable, as they are the endpoints of Hawking radiation. In string theory, some D branes are BPS.

- BPS states are important for understanding strong/weak coupling dualities, because they are distinguished by short multiplets, and multiplets can't change size as the coupling continuously changes from weak to strong.

Note. We can see how the Higgs mechanism would work in a supersymmetric field theory by looking at the structure of the multiplets. Without supersymmetry, a helicity ± 1 Poincare irrep "eats" a helicity 0 irrep to gain mass, forming a spin 1 irrep. Similarly, for $\mathcal{N} = 1$ a vector multiplet eats a chiral multiplet. Accounting for their CPT conjugates as well, this forms the $y = 1/2$ massive multiplet. For $\mathcal{N} = 2$ a vector multiplet again eats a chiral multiplet. Accounting for their CPT conjugates, this forms the $y = 0$ massive multiplet.

2.4 SUSY in Various Dimensions

Now we consider SUSY in various dimensions.

- In general, the SUSY generators Q^I (for $I = 1, \ldots, \mathcal{N}$) are taken to transform in the minimal spinor representation of $\mathfrak{so}(1, d-1)$. We define $\mathcal{N}$ to be the ratio of the number of real supercharges N_Q to the real dimension of the minimal spinor representation. However, it turns out that generally the important variable is N_Q, not $\mathcal{N}$.

- A summary of the degrees of freedom for low dimension is shown below.

d	spinors	N_Q	max R-sym	$\mathcal{N}_{\max}^{\text{SUSY}}$	$\mathcal{N}_{\max}^{\text{SUGRA}}$
1	$\mathcal{N}$ Majorana	$\mathcal{N}$	$SO(\mathcal{N})$	–	–
2	(n_L, n_R) Maj–Weyl	$n_L + n_R$	$SO(n_L) \times SO(n_R)$	$(8, 8)$	$(16, 16)$
3	$\mathcal{N}$ Majorana	$2\mathcal{N}$	$SO(\mathcal{N})$	8	16
4	$\mathcal{N}$ Weyl	$4\mathcal{N}$	$U(\mathcal{N})$	4	8
5	$\mathcal{N}$ Dirac	$8\mathcal{N}$	$Sp(\mathcal{N})$	2	4
6	(n_L, n_R) Weyl	$8(n_L + n_R)$	$Sp(n_L) \times Sp(n_R)$	$n_L + n_R = 2$	$n_L + n_R = 4$
7	$\mathcal{N}$ Dirac	$16\mathcal{N}$	$Sp(\mathcal{N})$	1	2
8	$\mathcal{N}$ Weyl	$16\mathcal{N}$	$U(\mathcal{N})$	1	2
9	$\mathcal{N}$ Majorana	$16\mathcal{N}$	$SO(\mathcal{N})$	1	2
10	(n_L, n_R) Maj–Weyl	$16(n_L + n_R)$	$SO(n_L) \times SO(n_R)$	$n_L + n_R = 1$	$n_L + n_R = 2$
11	$\mathcal{N}$ Majorana	$32\mathcal{N}$	$SO(\mathcal{N})$	0	1

  Note that in $d = 2, 6, 10$ there are two distinct "minimal" representations, which are not conjugate; hence $\mathcal{N}$ really must be described by two independent numbers. Similarly, in $d = 4$ (and others) the Weyl and Majorana spinors have the same real dimension, so one may take the supercharges to be either; the two are equivalent as real representations.

- As for $d = 4$, $\mathcal{N}_{\max}^{\text{SUSY}}$ is the largest amount of supersymmetry where we can have massless particles without requiring helicity $|\lambda| > 1$, while $\mathcal{N}_{\max}^{\text{SUGRA}}$ is the same for helicity $|\lambda| > 2$. **(typo in 11 row for number of degrees of freedom? shouldn't things be called Majorana?)**

- Looking at higher dimensions, there is no rigid supersymmetry beyond $d = 10$, and $\mathcal{N} = (1, 0)$ SYM in $d = 10$ is closely related to $\mathcal{N} = 4$ SYM in $d = 4$.

- Furthermore, there is no supergravity beyond $d = 11$. The $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ SUGRA theories in $d = 10$ are low-energy limits of type IIa and IIb superstring theory, respectively. The $\mathcal{N} = 1$ supergravity theory in $d = 11$ is thought to be the low-energy limit of M-theory.

Example. SUSY in $d = 2$. The Lorentz group is $SO(1, 1) \cong \mathbb{R}$, so all representations are one-dimensional and we may classify them by their $SO(1, 1)$ charge, which we call the spin. In the case $\mathcal{N} = (1, 1)$, there are right-moving and left-moving real supercharges $Q_\pm$ with spin $\pm 1/2$. The boost generator M transforms in the trivial representation of $SO(1, 1)$, while the translation operators P^μ decompose into spin ± 1. The algebra contains

$$[P^0, M] = P^1, \quad [P^1, M] = P^0, \quad [Q_\pm, M] = \pm Q_\pm$$

with all other commutators vanishing. As for the anticommutation relations between the SUSY charges, representation theory fixes

$$\{Q_+, Q_+\} = P_+, \quad \{Q_-, Q_-\} = P_-, \quad P_\pm = P^0 \pm P^1.$$

Next, since $\{Q_+, Q_-\}$ carries spin 0, it could contain M or a central charge. By the same argument as in $d = 4$, it can't contain M, so

$$\{Q_+, Q_-\} = Z.$$

For $\mathcal{N} = (2,0)$, the result is similar, but there would be no room to include central charges.

Example. SUSY in $d = 3$. In this case, the SUSY generators are two-component Majorana spinors. (An easy way to see this is to note that $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$, and the 2 of $\mathfrak{sl}(2,\mathbb{R})$ is manifestly real.) We take the gamma matrices to be

$$(\gamma^\mu)_\alpha{}^\beta = (-i\sigma^2, \sigma^1, \sigma^3).$$

Spinor indices may be raised and lowered using $\epsilon^{\alpha\beta}$ as in $d = 4$. We have

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho$$

where $\epsilon^{012} = 1$ and we use mostly positive signature. Then the spinor representation matrices are

$$\mathbf{M}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho}\gamma^\rho$$

which are pure imaginary as required, which implies

$$[M_{\mu\nu}, Q^I_\alpha] = \frac{i}{2}\epsilon_{\mu\nu\rho}(\gamma^\rho Q^I)_\alpha.$$

Since $2 \times 2 = 1 + 3$, the general anticommutator of the SUSY charges has the form

$$\{Q^I_\alpha, Q^J_\beta\} = c_0 \gamma^\mu_{\alpha\beta} P_\mu \delta^{IJ} + \epsilon_{\alpha\beta} Z^{IJ}$$

for an antisymmetric central charge Z^{IJ}, as $\gamma^\mu_{\alpha\beta}$ is symmetric in α and β. **(finish this)**

3 Superspace and Superfields

3.1 Superspace

So far we've only been working with particle states, but in quantum field theory we would like to understand them as excitations generated by fields. We hence turn to the problem of writing supersymmetric Lagrangians. This is difficult, because supersymmetry introduces strong constraints.

- Before supersymmetry, we constructed Lorentz-invariant Lagrangians using fields $\varphi(x^\mu)$ which were functions of coordinates x^μ in Minkowski space, transforming under a definite representation of the Lorentz group.
- For example, we could have introduced the four fields of the Dirac spinor as separate objects, but then Lorentz invariance would have strongly constrained the couplings. It's much more convenient to work with the Dirac spinor as one object.
- Similarly, in supersymmetry, we work with superfields $\Phi(X)$ which transform under a definite representation of the super-Poincare group. We will see this requires X to be in 'superspace', Minkowski space with extra Grassmann dimensions.
- Also note that, in contrast to the particle case, symmetries for the fields should ideally hold off-shell. That is, the Lagrangian must be SUSY-invariant independent of the equations of motion. We will find below that we must introduce auxiliary fields to achieve this.

To motivate superspace, we require the basics of group actions.

- Every Lie group G has a group manifold $\mathcal{M}_G$.
  - For $G = U(1)$, the elements are $g = e^{i\alpha}$ with $\alpha \in [0, 2\pi]$, so $M_G = U(1)$.
  - For $G = SU(2)$, the elements are
$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$
    which implies $\mathcal{M}_G = S^3$.
  - For $G = SL(2, \mathbb{C})$, we've already seen $\mathcal{M}_G = \mathbb{R}^3 \times S^3$.
- We can do the same reasoning for cosets G/H.
  - Consider $G/H = SU(2)/U(1) \cong SO(3)/SO(2)$. The $U(1)$ factor can be taken to be $\mathrm{diag}(e^{i\gamma}, e^{-i\gamma})$, which can be used to make the parameter α above real. Then $\mathcal{M}_{G/H} = S^2$.
  - More generally, $\mathcal{M}_{SO(n+1)/SO(n)} = S^n$.
  - We have Poincare/Lorentz $= \{\omega^{\mu\nu}, a^\mu\}/\{\omega^{\mu\nu}\} = \{a^\mu\} =$ Minkowski.
- This last result motivates us to define $\mathcal{N} = 1$ superspace as the coset
$$\text{super Poincare/Lorentz} = \{\omega^{\mu\nu}, a^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}}\}/\{\omega^{\mu\nu}\}.$$
  Explicitly, elements of the super Poincare group take the form
$$g = \exp\Big(i(\omega^{\mu\nu} M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \overline{\theta}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}})\Big)$$

where the θ^α and $\overline{\theta}_{\dot\alpha}$ transform like Weyl spinors and hence must be Grassmann numbers by spin-statistics. Note that this implies

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu, \quad [\theta^\alpha Q_\alpha, \overline{\theta}^{\dot\beta}\overline{Q}_{\dot\beta}] = 2\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}P_\mu.$$

Superspace is ordinary space augmented with the Grassmann dimensions θ^α, $\overline{\theta}_{\dot\alpha}$.

We now review properties of Grassmann numbers.

- For a single Grassmann number θ, an arbitrary function $f(\theta)$ can be expanded as

$$f(\theta) = f_0 + f_1\theta$$

and define $df/d\theta = f_1$. Integrals are defined as

$$\int d\theta = 0, \quad \int d\theta\,\theta = 1$$

which implies that the 'Dirac delta' is $\delta(\theta) = \theta$. Note that the integral is equal to the derivative.

- Now consider spinors of Grassmann numbers θ^α, $\overline{\theta}_{\dot\alpha}$. Their squares as defined, as earlier, by

$$\theta\theta = \theta^\alpha\theta_\alpha, \quad \overline{\theta\theta} = \overline{\theta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}$$

which gives the identities

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \quad \overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\theta\theta}.$$

- Derivatives are defined by

$$\partial_\alpha\theta^\beta \equiv \frac{\partial\theta^\beta}{\partial\theta^\alpha} = \delta_\alpha^\beta, \quad \overline{\partial}_{\dot\alpha}\overline{\theta}^{\dot\beta} \equiv \frac{\partial\overline{\theta}^{\dot\beta}}{\partial\overline{\theta}^{\dot\alpha}} = \delta_{\dot\alpha}^{\dot\beta}.$$

However, note that by raising and lowering indices, this implies that

$$\partial^\alpha\theta_\beta = -\delta_\beta^\alpha, \quad \overline{\partial}^{\dot\alpha}\overline{\theta}_{\dot\beta} = -\delta_{\dot\beta}^{\dot\alpha}.$$

In index-free notation we thus have

$$(\psi\partial)(\theta\chi) = \psi\chi, \quad (\overline{\psi\partial})(\overline{\theta\chi}) = -\overline{\psi\chi}.$$

- For multiple integrals, we have

$$\int d\theta^1 \int d\theta^2\,\theta^2\theta^1 = 1$$

but we also note that $\theta^2\theta^1 = (1/2)\theta\theta$. Then it is convenient to define

$$\int d^2\theta = \frac{1}{2}\int d\theta^1 \int d\theta^1, \quad d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}$$

which gives the simple result

$$\int d^2\theta\,\theta\theta = 1.$$

Similarly, we define

$$d^2\overline{\theta} = \frac{1}{4}d\overline{\theta}^{\dot\alpha}d\overline{\theta}^{\dot\beta}\epsilon_{\dot\alpha\dot\beta}, \quad \int d^2\overline{\theta}\,\overline{\theta\theta} = 1.$$

- Integration can again be related to differentiation,

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta, \quad \int d^2\overline{\theta} = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\overline{\partial}_{\dot\alpha}\overline{\partial}_{\dot\beta}.$$

3.2 The Scalar Superfield

We now define the $\mathcal{N}=1$ scalar superfield. First, we review ordinary scalar fields.

- Consider a scalar field $\varphi(x^\mu)$. It is an element of the function space $\mathcal{F}$ which is a representation of the Poincare group. Let $\mathcal{P}^\mu$ be the representation of P^μ on $\mathcal{F}$. Then
$$\varphi(x^\mu) \to \exp(ia_\mu \mathcal{P}^\mu)\varphi(x^\mu) = \varphi(x^\mu + a^\mu), \quad \mathcal{P}_\mu = -i\partial_\mu.$$
Similarly, we may define the action of $M^{\mu\nu}$ on $\mathcal{F}$. Since φ is a scalar, we have
$$\mathcal{M}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu).$$
If φ transformed in a nontrivial Lorentz representation, this expression would have extra terms, as the Lorentz transformation would act on the field indices.
- More generally, Lorentz transformations are defined as vector fields under spacetime, and the changes of fields under these transformations are given by Lie derivatives. In the case of a scalar, this is just the vector field acting on the scalar, as we see above.
- Upon quantization, φ is an operator in a Hilbert space, and
$$\varphi \to \exp(-ia_\mu P^\mu)\varphi \exp(ia_\mu P^\mu).$$
Comparing our two expressions, at first order in a_μ, the change in φ under translation is
$$\delta\varphi = i[\varphi, a_\mu P^\mu] = ia^\mu \mathcal{P}_\mu \varphi = a^\mu \partial_\mu \varphi.$$
Note that these results are not specific to scalar fields.

Next, we turn to the scalar superfield. More complicated superfields which are not scalar-valued can also appear, but the scalar superfield and restrictions of it will suffice for our purposes.

- An $\mathcal{N}=1$ scalar superfield is a function on superspace $S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot\alpha})$. Performing a Taylor expansion in the Grassmann variables, we get a finite number of terms,
$$\begin{aligned} S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot\alpha}) &= \varphi(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta\, M(x) + \overline{\theta\theta}\, N(x) \\ &\quad + (\theta\sigma^\mu\overline{\theta})V_\mu(x) + (\theta\theta)\,\overline{\theta\lambda}(x) + (\overline{\theta\theta})\,\theta\rho(x) + (\theta\theta)(\overline{\theta\theta})\,D(x). \end{aligned}$$
The scalar superfield contains fields that are not Lorentz scalars, such as ψ. The spinor fields are Grassmann; these Grassmann variables are independent of the superspace variables θ, and come in via the path integral measure.
- It's clear that the terms up to second order are the most general possible. We could write more third-order and fourth-order terms using σ^μ, but they would be redundant with our existing terms by Fierz identities.
- Since S is a scalar superfield, we know how Poincare transformations act on it, so we focus on 'supertranslations',
$$\begin{aligned} S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot\alpha}) &\to e^{-i(\epsilon Q + \overline{\epsilon}\overline{Q})} S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot\alpha}) e^{i(\epsilon Q + \overline{\epsilon}\overline{Q})} \\ &= e^{-i(\epsilon Q + \overline{\epsilon}\overline{Q})} e^{-i(x^\mu P_\mu + \theta Q + \overline{\theta}\overline{Q})} S(0,0,0) e^{i(x^\mu P_\mu + \theta Q + \overline{\theta}\overline{Q})} e^{i(\epsilon Q + \overline{\epsilon}\overline{Q})}. \end{aligned}$$
The second step is, at this point, a reasonable ansatz that we will verify holds.

- This expression may be simplified using the Baker-Campbell-Hausdorff formula
$$e^A e^B = e^{A+B+[A,B]/2+\dots}$$
where all higher-order terms are zero because in this case,
$$[A,B] = [x^\mu P_\mu + \theta Q + \overline{\theta Q}, \epsilon Q + \overline{\epsilon}\overline{Q}] = (-i(\epsilon\sigma^\mu\overline{\theta}) + i(\theta\sigma^\mu\overline{\epsilon}))P_\mu$$
which commutes with both A and B. Therefore,
$$S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) \to e^{-i((x+\delta x)P + (\theta+\delta\theta)Q + (\overline{\theta}+\delta\overline{\theta})\overline{Q})} S(0,0,0) e^{i((x+\delta x)P+(\theta+\delta\theta)Q+(\overline{\theta}+\delta\overline{\theta})\overline{Q})}$$
where
$$\delta x^\mu = -i(\epsilon\sigma^\mu\overline{\theta}) + i(\theta\sigma^\mu\overline{\epsilon}), \quad \delta\theta_\alpha = \epsilon_\alpha, \quad \delta\overline{\theta}_{\dot{\alpha}} = \overline{\epsilon}_{\dot{\alpha}}.$$
That is, a translation in superspace induces a translation in real space.

- Again, we can also think of φ classically as an element of a function space, where
$$S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) \to e^{i(\epsilon\mathcal{Q}+\overline{\epsilon}\overline{\mathcal{Q}})} S(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) = S(x^\mu - i(\epsilon\sigma^\mu\overline{\theta}) + i(\theta\sigma^\mu\overline{\epsilon}), \theta + \epsilon, \overline{\theta} + \overline{\epsilon})$$
where the second equality comes from our result above. Thus we have
$$\boxed{\mathcal{Q}_\alpha = -i\partial_\alpha - (\sigma^\mu)_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu, \quad \overline{\mathcal{Q}}_{\dot{\alpha}} = i\overline{\partial}_{\dot{\alpha}} + \theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu, \quad \mathcal{P}_\mu = -i\partial_\mu.}$$
We can then verify that $\mathcal{Q}_\alpha$ and $\overline{\mathcal{Q}_{\dot{\alpha}}}$ satisfy the supersymmetry algebra,
$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_{\dot{\alpha}}\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}}\mathcal{P}_\mu, \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = 0.$$
Note that this is sometimes phrased in terms of the variations $\delta_\epsilon = i\epsilon Q$, $\delta_{\overline{\epsilon}} = i\overline{\epsilon}\overline{Q}$, in which case
$$[\delta_\epsilon, \delta_{\overline{\epsilon}}] = -2(\epsilon\sigma^\mu\overline{\epsilon})P_\mu.$$

- Note that the 'extra' terms cancel out in the supertranslation operators,
$$\theta\mathcal{Q} + \overline{\theta\mathcal{Q}} = -i\theta^\alpha\partial_\alpha - i\overline{\theta}^{\dot{\alpha}}\overline{\partial}_{\dot{\alpha}}$$
which retroactively justifies our ansatz. By comparing our two expressions to first order in ϵ,
$$\delta S = i[S, \epsilon Q + \overline{\epsilon}\overline{Q}] = i(\epsilon\mathcal{Q} + \overline{\epsilon}\overline{\mathcal{Q}})S, \quad i(\epsilon\mathcal{Q} + \overline{\epsilon}\overline{\mathcal{Q}}) = \epsilon\partial - i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu - \overline{\epsilon}\overline{\partial} + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu.$$

- We now tabulate how each of the individual pieces transform.
$$\begin{aligned}
\delta\varphi &= \epsilon\psi + \overline{\epsilon\chi} \\
\delta\psi &= 2\epsilon M + \sigma^\mu\overline{\epsilon}(i\partial_\mu\varphi + V_\mu) \\
\delta\overline{\chi} &= 2\overline{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\varphi - V_\mu) \\
\delta M &= \overline{\epsilon}\overline{\lambda} - \frac{i}{2}\partial_\mu\psi\partial^\mu\overline{\epsilon} \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\overline{\chi} \\
\delta V_\mu &= \epsilon\sigma_\mu\overline{\lambda} + \rho\sigma_\mu\overline{\epsilon} + \frac{i}{2}(\partial^\nu\psi\sigma_\mu\overline{\sigma}_\nu\epsilon - \overline{\epsilon\sigma}_\nu\sigma_\mu\partial^\nu\overline{\chi}) \\
\delta\overline{\lambda} &= 2\overline{\epsilon}D + \frac{i}{2}(\overline{\sigma}^\nu\sigma^\mu\overline{\epsilon})\partial_\mu V_\nu + i\overline{\sigma}^\mu\epsilon\partial_\mu M \\
\delta\rho &= 2\epsilon D - \frac{i}{2}(\sigma^\nu\overline{\sigma}^\mu\epsilon)\partial_\mu V_\nu + i\sigma^\mu\overline{\epsilon}\partial_\nu N \\
\delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\overline{\lambda} - \rho\sigma^\mu\overline{\epsilon}).
\end{aligned}$$

Note that δD is a total derivative.

We now make some general remarks on superfields.

- A superfield is a function on superspace that transforms under supertranslations as $\delta S = i(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S$.
- If S_1 and S_2 are superfields, then so is $S_1 S_2$, because
$$\delta(S_1 S_2) = (\delta S_1)S_2 + S_1(\delta S_2) = (i(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S_1)S_2 + S_1(i(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}}))S_2.$$
But this is equal to $i(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S_1 S_2$, because $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ obey the Leibniz rule.
- Similarly, linear combinations of superfields are also superfields.
- The field $\partial_\mu S$ is a superfield, but $\partial_\alpha S$ is not, because
$$\delta(\partial_\alpha S) = \partial_\alpha(\delta S) = i\partial_\alpha(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S \neq i(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})(\partial_\alpha S)$$
because ∂_α and $\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}}$ do not commute. This makes sense because $\partial_\alpha S$ has terms only up to linear order in θ, while S has terms up to quadratic order.
- We can alternatively derive this with commutators/Poisson brackets. We have
$$\delta(\partial_\alpha S) = i[\partial_\alpha S, \epsilon Q + \bar{\epsilon}\overline{Q}] = i\partial_\alpha[S, \epsilon Q + \bar{\epsilon}\overline{Q}] = i\partial_\alpha(\epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}})S$$
as above. Here, Q is not a vector field; it is instead the Noether charge associated with supertranslation under $\mathcal{Q}$. Since the charge is integrated over superspace, $\partial_\alpha Q = 0$.
- Instead, we define the covariant derivatives
$$\mathcal{D}_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \overline{\mathcal{D}}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu$$
which satisfy
$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \overline{\mathcal{Q}}_{\dot{\beta}}\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{Q}}_{\dot{\beta}}\} = 0.$$
To verify these results, we need to use the fact that ϵ is anticommuting. Note that these covariant derivatives have nothing to do with gauge fields. The covariant derivatives are very similar to $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, but not quite the same.
- We can further verify that
$$\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\beta}}\} = 2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu, \quad \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{D}}_{\dot{\beta}}\} = 0, \quad [\mathcal{D}_\alpha, \epsilon \mathcal{Q} + \bar{\epsilon}\overline{\mathcal{Q}}] = 0$$
The last result shows that if S is a superfield, so is $\mathcal{D}_\alpha S$. The first result means that the connection $\nabla = (\partial_\mu, \mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}})$ on flat superspace has nontrivial torsion.

Note. Our derivations above are somewhat heuristic. On a deeper level, given a Lie group G with a subgroup H, there is an induced action of G on G/H by left multiplication; this is the action of the supersymmetry generators we found heuristically above. The SUSY covariant derivatives above are defined by the action of G on G/H by *right* multiplication, which is why they anticommute with the $\mathcal{Q}_\alpha$. The SUSY covariant derivatives (along with ∂_μ) define a connection on superspace, and the fact that $\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_\beta\} \neq 0$ indicates the connection has nontrivial torsion.

The scalar superfield is not an irreducible representation of supersymmetry, so it can be reduced.

- Suppose only the "body" φ is nonzero. Then under a supertranslation we pick up ψ and $\overline{\chi}$ terms unless $\partial_\mu \varphi = 0$, so we need φ to be constant, which is not interesting. We might also attempt to set only ψ nonzero, but then we pick up φ terms. Instead, we build our constraints using covariant derivatives.
- The chiral and antichiral superfield satisfy
$$\overline{\mathcal{D}}_{\dot{\alpha}} \Phi = 0, \quad \mathcal{D}_\alpha \overline{\Phi} = 0.$$
Intuitively, these only depend on θ and $\overline{\theta}$, respectively, so they contain only left-chiral and right-chiral spinor fields, respectively. As we'll see below, these generate the particles in a chiral multiplet.
- The vector or real superfield satisfies $V = V^\dagger$, where the dagger is a complex conjugate at the classical level. It is the SUSY analogue of a real vector gauge field. These generate the particles in a vector multiplet.
- The linear superfield L is a vector superfield satisfying $\mathcal{D}\mathcal{D}L = 0$. Since it contains the constraint $\partial_\mu V^\mu = 0$, it is the SUSY analogue of a conserved current.

Note. What about superspace beyond $\mathcal{N} = 1$ SUSY? Though such formalisms do exist, they are generally not useful. The problem is that for N_Q real supercharges, we require N_Q Grassmann directions and hence $2^{N_Q} = 2^{4\mathcal{N}}$ components for a general superfield. This is much larger than even a long multiplet, which has only $2^{2\mathcal{N}}$ states. The constraints required to eliminate this many degrees of freedom become quite complicated, and in some cases cannot be imposed consistently. For general dimension, a rule of thumb is that a superspace formalism is only useful for $N_Q \leq 4$.

3.3 Chiral and Vector Superfields

We now consider the chiral superfield in detail.

- For convenience, we define
$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$$
and consider $\Phi = \Phi(y(x, \theta, \overline{\theta}), \theta, \overline{\theta})$. The covariant derivative is
$$\overline{\mathcal{D}}_{\dot{\alpha}} \Phi = \overline{\partial}_{\dot{\alpha}} \Phi + \frac{\partial \Phi}{\partial y^\mu} \frac{\partial y^\mu}{\partial \overline{\theta}^{\dot{\alpha}}} + i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \Phi = \overline{\partial}_{\dot{\alpha}} \Phi - i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \Phi + i\theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \Phi = \overline{\partial}_{\dot{\alpha}} \Phi$$
where we picked up a minus sign from anticommuting a Grassmann derivative. That is, in the $(y, \theta, \overline{\theta})$ variables, the covariant derivatives act as
$$\overline{D}_{\dot{\alpha}} = \overline{\partial}_{\dot{\alpha}}, \quad \mathcal{D}_\alpha = \partial_\alpha + 2i(\sigma^\mu \overline{\theta})_\alpha \frac{\partial}{\partial y^\mu}.$$
- Therefore, a chiral superfield obeys $\overline{\partial}_{\dot{\alpha}} \Phi = 0$ when expressed in terms of y, θ, and $\overline{\theta}$, so
$$\Phi(y, \theta) = \varphi + \sqrt{2}\theta\psi + \theta\theta F$$
where we suppressed the position argument y of the fields on the right. Here, φ and F are complex scalar fields, and ψ is a left-chiral Weyl spinor.

- The field F is auxiliary, and hence determined by the others on-shell. Then on-shell there are 2 spin 0 degrees of freedom and 2 spin 1/2 degrees of freedom, obeying $n_B = n_F$.

- Note that off-shell, there are 4 bosonic degrees of freedom, since φ and F are complex, and 4 fermionic degrees of freedom. This is convenient, because it means SUSY is manifest even off-shell, and is essentially the reason we need the F field.

- Expressing Φ in terms of x^μ and Taylor expanding, we have

$$\boxed{\Phi(x,\theta,\overline{\theta}) = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\overline{\theta})\partial_\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\overline{\theta} - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\partial_\mu\partial^\mu\varphi}$$

  where we suppressed x-dependence. Note that this is not an approximation; higher-order terms in the Taylor expansion are just all identically zero.

- Under a supersymmetry transformation $\delta\Phi = i(\epsilon\mathcal{Q} + \overline{\epsilon}\overline{\mathcal{Q}})\Phi$, the components change as

$$\delta\varphi = \sqrt{2}\,\epsilon\psi, \quad \delta\psi = i\sqrt{2}\,\sigma^\mu\overline{\epsilon}\partial_\mu\varphi + \sqrt{2}\,\epsilon F, \quad \delta F = i\sqrt{2}\,\overline{\epsilon}\overline{\sigma}^\mu\partial_\mu\psi.$$

  In particular, note that δF is a total derivative, just as δD was for a general superfield. It is straightforward to check manually that this satisfies the supersymmetry algebra. Later on, the transformation rules for the anti-chiral multiplet will also be useful; they are

$$\delta\overline{\varphi} = \sqrt{2}\,\overline{\epsilon}\overline{\psi}, \quad \delta\psi = i\sqrt{2}\,\overline{\sigma}^\mu\epsilon\partial_\mu\overline{\varphi} + \sqrt{2}\,\overline{\epsilon}\overline{F}, \quad \delta\overline{F} = i\sqrt{2}\,\epsilon\sigma^\mu\partial_\mu\overline{\psi}.$$

- Note that the product of chiral superfields is also a chiral superfield, with

$$\Phi_1\Phi_2 = (\varphi_1 + \sqrt{2}\theta\psi_1 + \theta\theta F_1)(\varphi_2 + \sqrt{2}\theta\psi_2 + \theta\theta F_2)$$

  giving

$$\varphi' = \varphi_1\varphi_2, \quad \psi' = \psi_1\varphi_2 + \psi_2\varphi_1, \quad F' = F_1\varphi_2 + F_2\varphi_1 - \psi_1\psi_2.$$

  This is one of the key benefits of superfields; we may easily take products of them.

- More generally, any holomorphic function $f(\Phi)$ is also chiral, but $\overline{\Phi} = \Phi^\dagger$ is antichiral. The fields $\Phi^\dagger\Phi$ and $\Phi^\dagger + \Phi$ are real, but neither chiral nor antichiral.

- We can further constrain chiral superfields. For example, let X be a nilpotent chiral superfield, so $X^2 = 0$ and $\mathcal{D}_{\dot{\alpha}}X = 0$. Renaming some of the fields, we have

$$X(y,\theta) = x + \sqrt{2}\theta\psi_x + \theta\theta F_x$$

  and squaring this gives

$$X^2 = x^2 + 2\sqrt{2}\,x\theta\psi_x + (2xF_x - \psi_x^2)(\theta\theta) = 0.$$

  The final term vanishes if $x = \psi_x^2/2F_x$, and this makes the first two terms automatically vanish as well, because they are proportional to ψ_x^4 and ψ_x^3, and ψ_x is a two-component spinor. We see the scalar field is a 'composite' of the fermion. However, it is only well-defined if F_x is nonzero, which we will see indicates SUSY breaking.

- In the absence of a mass term, the chiral superfield corresponds to the particles of an $\mathcal{N} = 1$ chiral multiplet. With a mass, it corresponds to the particles of an $\mathcal{N} = 1$ massive multiplet with superspin $y = 0$.

Next, we turn to the vector superfield.

- The most general vector superfield has the form

$$V(x, \theta, \overline{\theta}) = C + i\theta\chi - i\overline{\theta}\overline{\chi} + \frac{i}{2}\theta\theta(M + iN) - \frac{i}{2}\overline{\theta\theta}(M - iN) + \theta\sigma^\mu\overline{\theta}V_\mu$$
$$+ i\theta\theta\,\overline{\theta}\left(-i\overline{\lambda} + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi\right) - i\overline{\theta\theta}\,\theta\left(i\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}\right) + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\left(D - \frac{1}{2}\partial_\mu\partial^\mu C\right)$$

  where we have shifted some fields with respect to their definitions in the general superfield for convenience. Off shell, there are eight bosonic components, as C, M, N, D, and V^μ are all real, and eight fermionic components, from the complex χ and λ.

- Just as in the non-supersymmetric case, we want to impose a gauge symmetry to get rid of the unwanted degrees of freedom in the vector. If Λ is a chiral superfield, then $i(\Lambda - \Lambda^\dagger)$ is a vector superfield with components

$$C = i(\varphi - \varphi^\dagger), \quad \chi = \sqrt{2}\,\psi, \quad \frac{1}{2}(M + iN) = F, \quad V_\mu = -\partial_\mu(\varphi + \varphi^\dagger), \quad \lambda = D = 0.$$

  We may define a generalized gauge transformation on vector superfields by

$$V \to V - \frac{i}{2}(\Lambda - \Lambda^\dagger).$$

  This generalizes the ordinary notion of a gauge transformation as it acts on V_μ by

$$V_\mu \to V_\mu + \partial_\mu \operatorname{Re}(\varphi) \equiv V_\mu - \partial_\mu\alpha.$$

- We may use this gauge freedom to remove some of the vector superfield components. In Wess–Zumino gauge, which we use exclusively, we set $C = \chi = M = N = 0$, giving

$$\boxed{V_{\mathrm{WZ}}(x, \theta, \overline{\theta}) = (\theta\sigma^\mu\overline{\theta})V_\mu + (\theta\theta)(\overline{\theta}\overline{\lambda}) + (\overline{\theta\theta})(\theta\lambda) + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})D.}$$

  This leaves only the usual gauge freedom of V_μ. We hence have a vector field which yields gauge bosons, spinor fields which yield their superpartners, and another auxiliary field D.

- Wess–Zumino gauge is not SUSY invariant; if we perform a SUSY transformation we must perform a further gauge transformation to return to Wess–Zumino gauge. This is like how some QED gauges are not Lorentz invariant.

- Powers of V_{WZ} are given by

$$V_{\mathrm{WZ}}^2 = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})V^\mu V_\mu$$

  with all higher powers equal to zero.

- Given the gauge symmetry above, the vector superfield corresponds to the massless particles of an $\mathcal{N}=1$ vector multiplet. In the supersymmetric analogue of the Higgs effect, a chiral superfield couples to the vector superfield, and in terms of the particles, the vector multiplet 'eats' the chiral multiplet to become a massive $y=1/2$ multiplet.

- Again, the number of degrees of freedom balance off-shell, since the gauge field V_μ has 3 bosonic degrees of freedom and the real scalar D has 1, so an auxiliary field is again required.

Next, we introduce the abelian field strength superfield.

- Recall that a complex scalar field φ and a $U(1)$ gauge field V_μ have the gauge symmetry

$$\varphi \to e^{iq\alpha}\varphi, \quad V_\mu \to V_\mu + \partial_\mu \alpha$$

  where α is a real-valued field that specifies the gauge transformation. Starting with a free φ field, we may minimally couple it to the gauge field by a covariant derivative,

$$D_\mu \varphi = \partial_\mu \varphi - iqA_\mu \varphi, \quad \mathcal{L} \supset D^\mu \varphi (D_\mu \varphi)^*.$$

  The kinetic term for the gauge field is written using the gauge invariant field strength

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu, \quad \mathcal{L} \supset \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

- Similarly, in supersymmetry, we let a chiral and vector superfield have the gauge symmetry

$$\Phi \to e^{iq\Lambda}\Phi, \quad V \to V - \frac{i}{2}(\Lambda - \Lambda^\dagger)$$

  where $\Lambda(x)$ is a chiral superfield that specifies the gauge transformation; this is necessary to ensure that Φ remains a chiral superfield upon gauge transforming it. Note the term

$$\Phi^\dagger \exp(2qV)\Phi$$

  is gauge invariant and can serve as an interaction term.

- For a generic superfield, the spinor-valued superfield

$$W_\alpha \equiv -\frac{1}{4}(\overline{\mathcal{D}\mathcal{D}}) D_\alpha S$$

  is a chiral superfield, because applying $\overline{\mathcal{D}}$ will give three powers of $\overline{\mathcal{D}}$. The three indices on the factors of $\overline{\mathcal{D}}$ must be totally antisymmetric, since the $\overline{\mathcal{D}}$ anticommute with each other, but this implies they are identically zero.

- In the case where S is the vector superfield V, the result

$$W_\alpha = -\frac{1}{4}(\overline{\mathcal{D}\mathcal{D}}) D_\alpha V$$

  is also invariant under generalized gauge transformations; it is called the field strength superfield. We can similarly define an anti-chiral field strength superfield $\overline{W}_{\dot\alpha}$.

- To get an explicit expression, it's most useful to work in terms of y rather than x, giving

$$\boxed{W_\alpha(y,\theta) = \lambda_\alpha + \theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} - i(\theta\theta)(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\overline{\lambda}^{\dot\beta}.}$$

  Since W is invariant, the fields λ, D, and $F_{\mu\nu}$ are all separately gauge invariant. Note that since we have a spinor-valued superfield, the coefficient of θ_β is a bispinor, which naturally decomposes into a scalar (the field D) and a self-dual two-form (the field $F_{\mu\nu}$).

- We have the identity

$$D^\alpha W_\alpha = \overline{D}_{\dot\alpha}\overline{W}^{\dot\alpha}$$

  which contains the Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0$ as a subset.

- This result can also be generalized to non-abelian gauge fields, in which case we pick up extra terms from the structure constants, and our gauge invariant fields become gauge covariant.

4 Supersymmetric Lagrangians

4.1 $\mathcal{N} = 1$ Supersymmetry

Now we write supersymmetric Lagrangians, beginning with the simplest case of a chiral superfield.

- A theory with Lagrangian $\mathcal{L}(\Phi)$ is supersymmetric if the variation $\delta\mathcal{L}$ under supersymmetry transformations is a total derivative. Now recall that for a general scalar superfield S,

$$S \supset (\theta\theta)(\overline{\theta\theta})D, \quad \delta D = \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\overline{\lambda} - \rho\sigma^\mu\overline{\epsilon})$$

  and for a general chiral superfield Φ,

$$\Phi \supset (\theta\theta)F, \quad \delta F = i\sqrt{2}\,\overline{\epsilon\sigma}^\mu\partial_\mu\psi.$$

  Therefore, the Lagrangian is supersymmetric if it is built from the D terms of superfields and the F terms of chiral superfields. This is not surprising, since the terms in $\mathcal{Q}_\alpha$ and $\overline{\mathcal{Q}}_{\dot\alpha}$ that multiply by Grassmann numbers come with factors of ∂_μ.

- Note that for the Lagrangian to be real, the D terms come from real superfields and the F terms of chiral superfields are paired with the $\overline{F}$ terms of antichiral superfields. In particular, for a theory with a single chiral superfield Φ, the most general possibility is

$$\mathcal{L} = K(\Phi, \Phi^\dagger)|_D + (W(\Phi)|_F + \text{h.c.})\,.$$

  Here the Kahler potential K is a real function of Φ and $\Phi^\dagger$, the conjugate $\Phi^\dagger$ is an anti-chiral superfield, and the superpotential W is a holomorphic function of Φ, and hence a chiral superfield. Here $K|_D$ just means the coefficient of $(\theta\theta)(\overline{\theta\theta})$ in K.

- The action may thus be written as a superspace integral

$$S = \int d^4x \int d^4\theta\, K + \int d^4x \left(\int d^2\theta\, W + \text{h.c.}\right).$$

  Note that not every term is integrated over all of superspace. This is perfectly acceptable and also occurs in string theory, where objects may be confined to branes.

- Next, we perform dimensional analysis to determine renormalizability. Ultimately, we just have a complicated collection of scalar and fermion fields, which must have the usual dimensions,

$$[\varphi] = 1, \quad [\psi] = \frac{3}{2}.$$

  On the other hand, the chiral superfield is

$$\Phi = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F, \quad [\Phi] = 1, \quad [\theta] = -\frac{1}{2}, \quad [d\theta] = \frac{1}{2}, \quad [F] = 2.$$

  Here F does not have the usual dimensions of a scalar field, because it is an auxiliary field.

- The Kahler potential and superpotential take the form

$$K \supset (\theta\theta)(\overline{\theta\theta})K_D, \quad W \supset (\theta\theta)W_F$$

and renormalizability requires the operators in K_D and W_F to have dimensions at most 4, so

$$[K] \leq 2, \; [W] \leq 3.$$

Therefore, the Kahler potential is at most quadratic and the superpotential is at most cubic. However, the terms $\Phi + \Phi^\dagger$ and $\Phi\Phi + \text{h.c.}$ do not have D terms, so the most general Kahler potential is $K = \Phi^\dagger\Phi$, where we rescaled to remove the coefficient.

- For one chiral superfield, the general Lagrangian is known as the Wess–Zumino model,

$$\mathcal{L}_{\text{WZ}} = \Phi^\dagger\Phi|_D + (W(\Phi)|_F + \text{h.c.})\,.$$

Evaluating the Kahler potential term is straightforward and yields the kinetic terms along with FF^*. The superpotential provides interactions; to evaluate it, we perform a Taylor expansion in the Grassmann variables,

$$W(\Phi) = W(\varphi) + (\Phi - \varphi)\frac{\partial W}{\partial \varphi} + \frac{1}{2}(\Phi - \varphi)^2\frac{\partial^2 W}{\partial \varphi^2}, \quad \frac{\partial W}{\partial \varphi} \equiv \left.\frac{\partial W}{\partial \Phi}\right|_{\Phi = \varphi}$$

where the linear term contributes $\theta\theta F$ and the quadratic term contributes $(\theta\psi)(\theta\psi)$. Then

$$\mathcal{L}_{\text{WZ}} = \partial^\mu\varphi^*\partial_\mu\varphi - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + FF^* + \left(\frac{\partial W}{\partial \varphi}F + \text{h.c.}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial \varphi^2}\psi\psi + \text{h.c.}\right).$$

Historically this was the first nontrivial four-dimensional supersymmetric model, and it was originally written without the benefit of superspace and superfields.

- The portion of the Lagrangian that depends on F is

$$\mathcal{L}_F = FF^* + \frac{\partial W}{\partial \varphi}F + \frac{\partial W^*}{\partial \varphi^*}F^*.$$

There is no kinetic term, confirming F is an auxiliary field. Since it is quadratic, it is straightforward to eliminate F. Setting $\delta S_F/\delta F = 0$, we find

$$F^* + \frac{\partial W}{\partial \varphi} = 0, \quad F + \frac{\partial W^*}{\partial \varphi^*} = 0.$$

Substituting this back in,

$$\mathcal{L}_F = -\left|\frac{\partial W}{\partial \varphi}\right|^2 \equiv -V_F(\varphi).$$

That is, these terms simply yield a positive semi-definite scalar potential for φ.

- We can eliminate the auxiliary field F from the Wess–Zumino model by plugging in its equations of motion. The cost of doing this is that the Lagrangian becomes SUSY invariant only on-shell; thus we prefer to keep it explicit if we're not doing practical calculations.

- We may regard our $\mathcal{N}=1$ Lagrangian as a special $\mathcal{N}=0$ Lagrangian. Field redefinitions can be used to remove the linear term in W, and the constant term doesn't matter, so
$$W = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.$$
We note that both the complex scalar φ and spinor ψ have mass m, and the Yukawa coupling is the same as the scalar self-coupling, $\mathcal{L} \supset g(\varphi\psi\psi) + g^2|\varphi|^4$. As shown earlier, this ensures that divergences cancel in perturbation theory.

- We now generalize to multiple chiral superfields. We have a Kahler potential $K(\Phi^i, \Phi^{j\dagger})$ and superpotential $W(\Phi^i)$. Expanding about $\Phi^i = \varphi^i$,
$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial\varphi^i \partial\varphi^{\bar{j}*}} \equiv \partial_i \partial_{\bar{j}} K$$
where a bar denotes a conjugated field. Here, $K_{i\bar{j}}$ may be regarded as a metric in a space with coordinates φ^i which is a complex manifold, specifically a Kahler manifold because the metric can be derived by differentiating a potential. For a renormalizable theory this is rather trivial, as $K_{i\bar{j}}$ is constant, but if we allow the Kahler potential to be arbitrary (e.g. viewing the theory as an effective theory) we can get more complicated manifolds.

- The kinetic terms become
$$\mathcal{L} \supset K_{i\bar{j}}\left(\partial^\mu \varphi^{\bar{j}*}\partial_\mu \varphi^i - i\overline{\psi}^{\bar{j}}\overline{\sigma}^\mu \partial_\mu \psi^i + F^i F^{\bar{j}*}\right) + \ldots$$
where a general Kahler potential produces extra fermion interaction terms. We say the Kahler potential is canonical if $K_{i\bar{j}}$ is diagonal and constant, which can be achieved by a $U(n)$ field redefinition in the renormalizable case.

- Restricting to renormalizable theories, the superpotential is expanded as before, resulting in
$$\mathcal{L}_F = K_{i\bar{j}} F^i F^{\bar{j}*} + (\partial_i W) F^i + (\partial_{\bar{i}} W^*) F^{\bar{i}*}.$$
Varying with respect to F, the auxiliary field is
$$K_{i\bar{j}} F^{\bar{j}*} + \partial_i W = 0, \quad K_{i\bar{j}} F^i + \partial_{\bar{j}} W^* = 0.$$
Plugging these results in gives a contribution to the scalar potential of
$$V_F = K_{i\bar{j}} F^i F^{\bar{j}*} = K^{i\bar{j}} \partial_i W \partial_{\bar{j}} W^*$$
where the Kahler metric with raised indices is the inverse of the original Kahler metric.

Note. Keeping track of the R-symmetry. The $U(1)$ R-symmetry in such theories takes the form
$$\Phi^i \to e^{ir_i\alpha}\Phi^i$$
where $r_i = R[\Phi^i]$ is the $U(1)_R$ charge of the field Φ^i, and $R[\overline{\Phi}^{\bar{i}}] = -r_i$. By definition, we have
$$R[Q_\alpha] = -1, \quad R[\overline{Q}_{\dot{\alpha}}] = 1$$

which implies that the superspace coordinates are charged as well,

$$R[\theta] = 1, \quad R[\bar{\theta}] = -1, \quad R[d\theta] = -1, \quad R[d\bar{\theta}] = 1.$$

This implies that the components of a chiral superfield have different R-charges,

$$R[\varphi^i] = r_i, \quad R[\psi^i] = r_i - 1, \quad R[F^i] = r_i - 2.$$

The Kahler potential is invariant under $U(1)_R$ provided that it does not mix fields of different R-charges. The superpotential is only invariant if $R[W] = 2$, and hence it generically breaks the R-symmetry. We also note that the R-symmetry may be ambiguous, as we may combine it with flavor symmetries which rotate the Φ^i individually.

Next, we turn to the vector superfield Lagrangian, i.e. the theory of “super QED”.

- We can deduce QED by demanding a local $U(1)$ symmetry for a complex scalar field φ, parametrized by a scalar field α. To get super QED we replace the scalar fields with chiral superfields, with the gauge transformation

  $$\Phi \to \exp(iq\Lambda)\Phi.$$

  As in QED, the naive kinetic term $K = \Phi^\dagger \Phi$ is no longer gauge invariant, but

  $$K = \Phi^\dagger \exp(2qV)\Phi, \quad V \to V - \frac{i}{2}(\Lambda - \Lambda^\dagger)$$

  is gauge invariant, as shown above. The kinetic term for the vector superfield/gauge field V is

  $$\mathcal{L}_{\text{kin}} = f(\Phi)(W^\alpha W_\alpha)|_F + \text{h.c.}$$

  where renormalizability requires the ‘gauge kinetic function’ f to be a constant, $f = \tau$.

- A new feature of super QED is a new gauge-invariant term, the Fayet–Iliopoulos (FI) term

  $$\mathcal{L}_{\text{FI}} = \xi V|_D = \frac{1}{2}\xi D$$

  where ξ is a constant. The FI term only appears for an abelian gauge theory, because the gauge field is not charged under $U(1)$. More generally the gauge field would be charged, which would make D charged and the FI term not gauge invariant.

- Therefore, the renormalizable Lagrangian of super QED is

  $$\mathcal{L} = (\Phi^\dagger \exp(2qV)\Phi)\big|_D + \left(\left(W(\Phi) + \frac{1}{4}W^\alpha W_\alpha\right)\bigg|_F + \text{h.c.}\right) + \xi V\big|_D$$

  where we have set $\tau = 1/4$ to get a canonically normalized photon field.

- Note that for the superpotential $W(\Phi)$ to be gauge invariant, it must contain terms like $\Phi_1 \cdots \Phi_n$ where the charges of the terms in the product add to zero. If there is only one charged Φ field, the superpotential must vanish.

- Next, we write out the first term explicitly. Taking Wess–Zumino gauge and Taylor expanding,

$$\exp(2qV) = 1 + 2qV + 2q^2V^2$$

where all higher terms vanish. We thus find

$$(\Phi^\dagger \exp(2qV)\Phi)\big|_D = F^*F + \partial_\mu\varphi\partial^\mu\varphi^* - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + qV^\mu(-\overline{\psi}\overline{\sigma}_\mu\psi + i\varphi^*\partial_\mu\varphi - i\varphi\partial_\mu\varphi^*) \\ + \sqrt{2}q(\varphi\overline{\lambda\psi} + \varphi^*\lambda\psi) + q^2(D + qV_\mu V^\mu)|\varphi|^2.$$

We could make this manifestly gauge invariant by grouping terms into covariant derivatives.

- Next, we consider the gauge kinetic term. We have

$$W^\alpha W_\alpha|_F = D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} - \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

where, to get the $F_{\mu\nu}$ terms, we used the identity

$$\operatorname{tr}\sigma^{\mu\nu}\sigma^{\kappa\tau} = \frac{1}{2}(\eta^{\mu\kappa}\eta^{\nu\tau} - \eta^{\mu\tau}\eta^{\nu\kappa} + i\epsilon^{\mu\nu\kappa\tau}).$$

- Then the gauge kinetic term is

$$\frac{1}{4}W^\alpha W_\alpha\bigg|_F + \text{h.c.} = \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\overline{\lambda}.$$

More generally, we could take τ complex, yielding an $F\tilde{F}$ term. However, this term is a total derivative, so it makes no difference perturbatively.

- Collecting the terms involving D, we have

$$\mathcal{L}_D = qD|\varphi|^2 + \frac{1}{2}D^2 + \frac{1}{2}\xi D$$

so D is an auxiliary field like F. Setting $\delta S_D/\delta D = 0$ gives

$$D = -\frac{\xi}{2} - q|\varphi|^2.$$

Substituting this back in gives

$$\mathcal{L}_D = -\frac{1}{2}D^2 = -\frac{1}{2}\left(\frac{\xi}{2} + q|\varphi|^2\right)^2 \equiv -V_D(\varphi)$$

so we again get a contribution to the scalar potential. The total scalar potential is

$$V(\varphi) = V_F(\varphi) + V_D(\varphi) = \left|\frac{\partial W}{\partial\varphi}\right|^2 + \frac{1}{2}\left(\frac{\xi}{2} + q|\varphi|^2\right)^2 \geq 0.$$

We see that the FI term could be responsible for spontaneous symmetry breaking.

- Finally, the general action can be written as a superspace integral as

$$S[K, W, f, \xi] = \int d^4x \int d^4\theta\,(K + \xi V) + \int d^4x\left(\int d^2\theta\,(W + fW^\alpha W_\alpha) + \text{h.c.}\right).$$

We will not consider the non-abelian case, where there are many complications.

- The results easily generalize for more fields. Assuming the gauge group remains $U(1)$, we don't get more vector superfields, but we can have multiple chiral superfields Φ^i. We find

$$\mathcal{L}_D = qDK_{i\bar{j}}\varphi^i\varphi^{\bar{j}*} + \frac{1}{2}D^2 + \frac{1}{2}\xi D, \quad D = -\frac{\xi}{2} - qK_{i\bar{j}}\varphi^i\varphi^{\bar{j}*}$$

so the total scalar potential is

$$V(\varphi^i) = K_{i\bar{j}}F^iF^{\bar{j}*} + \frac{1}{2}D^2 = K^{i\bar{j}}\partial_i W\partial_{\bar{j}}W^* + \frac{1}{2}\left(\frac{\xi}{2} + qK_{i\bar{j}}\varphi^i\varphi^{\bar{j}*}\right)^2.$$

- To get the supersymmetric analogue of ordinary QED, with a single Dirac fermion, we require two chiral superfields Φ_+ and Φ_- with opposite charges, corresponding to the electron and positron. The Dirac mass term comes from the superpotential $W = m\Phi_+\Phi_-$. A single massive chiral superfield instead corresponds to a Majorana fermion, which cannot be charged.

4.2 The Wess–Zumino Model

In this section, we consider the renormalization of the Wess–Zumino model in detail.

- We take the superpotential to have the standard form given above. To establish notation, the Lagrangian has kinetic terms

$$\mathcal{L}_0 = -\partial_\mu\overline{\phi}\partial^\mu\phi - m^2\overline{\phi}\phi - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi - \frac{m}{2}(\psi\psi + \overline{\psi\psi})$$

where we are using $(-+++)$ signature, so the propagators are: where the momentum p flows to the right.

- The interaction terms are

$$\mathcal{L}_{\text{int}} = -\lambda^2\overline{\phi}^2\phi^2 - m\lambda(\overline{\phi}^2\phi + \overline{\phi}\phi^2) - \lambda(\psi\psi\phi + \overline{\psi\psi\phi}).$$

For the scalar, this leads to cubic and quartic vertices of $-im\lambda$ and $-i\lambda^2$ respectively, as well as the Yukawa interactions:

- We could have also treated the fermion mass as an interaction term, in which case only $\langle\psi\psi\rangle$ and $\langle\overline{\psi\psi}\rangle$ would be nonzero, and we would have an interaction term $-im$ which reversed the chirality.

- At one loop, the total scalar tadpole vanishes, as the two diagrams cancel: In fact, this cancellation holds at all orders in perturbation theory, and it implies that vevs are not renormalized.

- Next, consider the scalar self-energy at zero external momentum. This directly determines the coefficient of the mass term $\overline{\phi}\phi$ in the effective action, and three diagrams contribute: The total contribution is

$$(-im\lambda)^2\int đq\left(\frac{-i}{q^2+m^2}\right)^2 - (-i\lambda)^2\frac{1}{2}\int đq\,\frac{\text{tr}(-\sigma^\mu\overline{\sigma}^\nu)q_\mu q_\nu}{(q^2+m^2)^2} + (-i\lambda)^2\int đq\,\frac{-i}{q^2+m^2} = 0.$$

Hence the mass term isn't renormalized either, and this again holds to all orders in perturbation theory. One can check the same holds for all of the vertices.

- We could also formulate the perturbation theory with the auxiliary field,

$$\mathcal{L}_0 = -\partial_\mu\overline{\phi}\partial^\mu\phi - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi + \overline{F}F + m(F\phi + \overline{F\phi} - \frac{1}{2}\psi\psi - \frac{1}{2}\overline{\psi\psi})$$

which has a simpler set of interactions,

$$\mathcal{L}_{\text{int}} = \lambda(F\phi^2 + \overline{F\phi}^2 - \psi\psi\phi - \overline{\psi\psi\phi}).$$

- To compute the scalar propagators, write the kinetic terms as $(\phi, F)M(\phi, F)^T$ and invert M, to find

$$\langle\overline{\phi}\phi\rangle = \frac{-i}{p^2+m^2}, \quad \langle\overline{F}F\rangle = \frac{ip^2}{p^2+m^2}, \quad \langle\phi F\rangle = \langle\overline{\phi F}\rangle = \frac{im}{p^2+m^2}.$$

In particular, there is a nontrivial, though unusual propagator for F.

- Upon computation of the effective action, we find no effects at one loop except for the same field strength renormalization factor

$$Z_\Phi \equiv Z_\phi = Z_\psi = Z_F.$$

This continues at all orders in perturbation theory. The only effect is the anomalous dimension $\gamma = d\log Z_\Phi/d\log\mu$. If we account for renormalization effects by working in terms of the renormalized fields $\Phi_R = Z_\Phi^{1/2}\Phi$, and γ is given, then we can compute the RG flow of the renormalized couplings exactly.

4.3 Non-Renormalization Theorems

We have seen that theories of chiral and vector superfields are determined by the functions K, W, f, and the parameter ξ. We now investigate how they are renormalized in general.

- In 1977, Grisaru, Siegel, and Rocek showed using "supergraphs" that, except for one-loop corrections to f, quantum corrections only come in the form

$$\int d^4x \int d^4\theta \, \ldots.$$

Then W and ξ are not renormalized in perturbation theory at all, while K is.

- In 1993, Seiberg used symmetry and holomorphicity arguments to establish this result nonperturbatively in a simple and elegant way; we will follow this proof here. The intuition is that since the superpotential is holomorphic, it is determined by its singularities and its asymptotics; this allows us to pin it down in general using its weak coupling limit.

- In order to keep track of symmetries, we introduce spurion superfields

$$X = (x, \psi_x, F_x), \quad Y = (y, \psi_y, F_y)$$

so the action becomes

$$S = \int d^4x \int d^4\theta\,(K + \xi V) + \int d^4x \left(\int d^2\theta\,(YW + XW^\alpha W_\alpha) + \text{h.c.}\right).$$

Here we note that the integrand of the $d^2\theta$ integral is holomorphic. These spurion fields have no dynamics; they are just a way to rewrite numerical coupling constants in the action to make symmetries manifest. We might think of them as being very heavy fields with fixed vevs.

- Specifically, the action has a $U(1)_R$ symmetry with charges

Φ_i	V	X	Y	θ	$\bar{\theta}$	W^α
0	0	0	2	1	-1	1

  where we note that if $\theta \to e^{i\alpha}\theta$, then $d\theta \to e^{-i\alpha} d\theta$ because $\int d\theta\, \theta = 1$, and the charge for W_α can be found from its definition in terms of covariant derivatives, using $\partial_\theta \to e^{-i\alpha}\partial_\theta$.

- The action also has a shift symmetry,

$$X \to X + ir, \quad r \in \mathbb{R}$$

  because the contribution of X to the action is

$$XW^\alpha W_\alpha \supset \mathrm{Re}(x) F_{\mu\nu}F^{\mu\nu} + \mathrm{Im}(x) F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

  and hence the shift contributes a total derivative, which does not affect perturbation theory. This symmetry is also called a Peccei-Quinn symmetry, making X an 'axion-like field'.

- Now consider the Wilsonian action S_Λ attained by integrating out all degrees of freedom above Λ. We must have

$$S_\Lambda = \int d^4x \int d^4\theta \, \big(J(\Phi, e^V, X, Y, \mathcal{D}, \ldots) + \xi(X,Y)V\big) + \int d^4x \int d^2\theta \, H(\Phi, X, Y, W^\alpha) + \text{h.c.}$$

  where H is a holomorphic function, and J and ξ are not.

- By $U(1)_R$ invariance we must have

$$H = Yh(X,\Phi) + g(X,\Phi)W^\alpha W_\alpha.$$

  Moreover, we must still have invariance under shifts in X. Hence the only term involving X that is allowed takes the form $XW^\alpha W_\alpha$, so

$$H = Yh(\Phi) + (\alpha X + g(\Phi))W^\alpha W_\alpha.$$

  Now, in the limit $Y \to 0$, we must have $h(\Phi) = W(\Phi)$, because any higher order corrections to $h(\Phi)$ would be higher order in Y and hence negligible. Thus we must have $h(\Phi) = W(\Phi)$ for all Y, so the superpotential is not renormalized!

- We claim the gauge kinetic function is only renormalized at one loop. Since the gauge kinetic term appears as $XW^\alpha W_\alpha$, the gauge field propagator is proportional to $1/x$ and the three-point gauge vertex is proportional to x. **(don't understand)** Then the number of powers of x at L loops is $1 - L$, so αX is the tree-level contribution and $g(\Phi)$ is the one-loop contribution. In practice, this means that one can compute divergent higher-loop corrections for the gauge kinetic function, but they all miraculously cancel.

- We cannot constrain the Kahler potential nearly as much, because it is not holomorphic. However, the FI term must be a constant to maintain gauge invariance under
$$V \to V + i(\Lambda - \Lambda^\dagger).$$
Moreover, the contributions correcting ξ are proportional to $\sum_i q_i$ where the q_i are the $U(1)$ charges. This vanishes if the gravitational anomaly vanishes, so ξ is not renormalized.

Note. The above derivation is a bit complicated; it's easier to see the key ideas in the Wess–Zumino model. Take the superpotential at scale μ_0 to be
$$W_{\mu_0} = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \equiv \mu_0 \frac{\tilde{m}}{2} + \frac{\lambda}{3}\Phi^3$$
where we defined the dimensionless coupling $\tilde{m}$. The free theory at $W = 0$ has a $U(1)$ rotation symmetry for Φ and a $U(1)_R$ symmetry, with charges
$$\Phi\colon (1,1), \quad \tilde{m}\colon (-2,0), \quad \lambda\colon (-3,-1).$$
Then the most general allowed for the effective superpotential at scale μ is
$$W_\mu = \mu\tilde{m}\Phi^2 f\left(\frac{\lambda\Phi}{\mu\tilde{m}}, \frac{\mu}{\mu_0}\right)$$
where f is holomorphic in its first argument. Since the limit $\lambda \to 0$ is regular, we can expand it in a Taylor series,
$$W_\mu = \sum_{n=0}^{\infty} c_n(\mu/\mu_0)\frac{\lambda^n}{(\mu\tilde{m})^{n-1}}\Phi^{n+2}.$$
Now we also need a regular $\tilde{m} \to 0$ limit, which means terms with $n > 1$ are disallowed. Then
$$W_\mu = c_0(\mu/\mu_0)\mu\tilde{m}\Phi^2 + c_1(\mu/\mu_0)\lambda\Phi^3.$$
But now we can take the weak coupling limit: we know that in the limit $\lambda \to 0$ no nontrivial renormalization occurs at all, so $c_0 = 1/2$ and $c_1 = 1/3$. Then
$$W_\mu = \mu\frac{\tilde{m}(\mu)}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3$$
which shows that the superpotential is not renormalized.

4.4 Extended Supersymmetry

Next, we briefly look at extended supersymmetry. We will write the results in $\mathcal{N} = 1$ language, i.e. the actions will just be $\mathcal{N} = 1$ actions with constraints.

- The simplest case in $\mathcal{N} = 2$ is the vector multiplet, whose particles are created by a massless chiral superfield Φ and massless vector superfield V, just as in super QED.
- Here, the $\mathcal{N} = 2$ supersymmetry imposes the constraint $W = 0$, ensuring Φ is massless, and
$$f(\Phi) = \frac{\partial^2 \mathcal{F}}{\partial \Phi^2}, \quad K(\Phi, \Phi^\dagger) = \frac{1}{2i}\left(\Phi^\dagger \exp(2V)\frac{\partial F}{\partial \Phi} - \text{h.c.}\right)$$
where $\mathcal{F}(\Phi)$ is a holomorphic function called the prepotential, where $\mathcal{F}(\Phi) = \Phi^2$ at tree level.

- It can be shown that $\mathcal{F}(\Phi)$ only receives one-loop corrections in perturbation theory. It also receives nonperturbative corrections which can be written in terms of an "instanton expansion" $\sum_k a_k \exp(-kc/g^2)$ found by Seiberg and Witten in 1994.
- There are other combinations of fields that produce $\mathcal{N}=2$ multiplets, but they are much more complicated.
- In $\mathcal{N}=4$, we consider the vector multiplet, which consists of an $\mathcal{N}=2$ vector multiplet and an $\mathcal{N}=2$ hypermultiplet. Here there are no free functions at all, only a single free parameter

$$f = \tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}.$$

  The theory is finite, i.e. has no UV divergences, and the beta function vanishes, yielding conformal invariance. This is the theory of $\mathcal{N}=4$ super Yang–Mills.
- The AdS/CFT correspondence relates a gravitational theory in AdS space to a conformal field theory without gravity in one fewer dimension. The prime example of this correspondence is between AdS in five dimensions and $\mathcal{N}=4$ super Yang–Mills in four dimensions.

5 The MSSM

5.1 SUSY Breaking

We now review the basics of supersymmetry breaking.

- Classically, suppose that fields transform under a symmetry as

$$\varphi_i \to \exp(i\alpha^a T^a)_i^{\ j}\varphi_j, \quad \delta\varphi_i = i\alpha^a (T^a)_i^{\ j}\varphi_j.$$

  The symmetry is said to be broken if the vacuum is not invariant,

$$\alpha^a (T^a)^i{}_j (\varphi_{\text{vac}})_j \neq 0.$$

- For example, for a complex field $\varphi = \rho e^{i\theta}$ with a $U(1)$ internal symmetry,

$$\delta\varphi = i\alpha\varphi, \quad \delta\rho = 0, \quad \delta\theta = \alpha.$$

  When the vacuum satisfies $\langle\rho\rangle \neq 0$, the symmetry is broken, and θ is a Goldstone boson.

- Using the SUSY commutation relations, note that

$$(\overline{\sigma}^\nu)^{\dot\beta\alpha}\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\overline{\sigma}^\nu)^{\dot\beta\alpha}(\sigma^\mu)_{\alpha\dot\beta}P_\mu = 4\eta^{\mu\nu}P_\mu = 4P^\nu.$$

  In particular, taking the $\nu = 0$ component, we have

$$(\overline{\sigma}^0)^{\dot\beta\alpha}\{Q_\alpha, \overline{Q}_{\dot\beta}\} = \sum_{\alpha=1}^{2}(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha) = 4P^0 = 4E.$$

  Since the left-hand side is positive definite, $E \geq 0$ for any state.

- In particular, consider the vacuum state $|\Omega\rangle$. If the vacuum is SUSY invariant, $Q_\alpha|\Omega\rangle = 0$, then the left-hand side vanishes and $E = 0$. If SUSY is broken, then the vacuum energy is positive.

- This might seem a bit strange, because we are used to the vacuum energy being indefinite in quantum field theory; for example, we change it by normal ordering the Hamiltonian. The difference is that the SUSY commutation relations involve the Hamiltonian itself, $H \sim |Q|^2$. Hence requiring a SUSY algebra at the quantum level constrains the operator ordering.

- As before, SUSY breaking is caused by fields acquiring vevs that are not SUSY invariant. Recall that in the case of a chiral superfield Φ,

$$\delta\varphi = \sqrt{2}\epsilon\psi, \quad \delta\psi = \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\overline{\epsilon}\partial_\mu\varphi, \quad \delta F = i\sqrt{2}\overline{\epsilon}\overline{\sigma}^\mu\partial_\mu\psi.$$

  The field ψ cannot have a vev, as this would violate Lorentz invariance. Similarly, we must have $\langle\partial_\mu\varphi\rangle = 0$. Then the transformation reduces to

$$\delta\langle\varphi\rangle = \delta\langle F\rangle = 0, \quad \delta\langle\psi\rangle = \sqrt{2}\epsilon\langle F\rangle.$$

  Hence, SUSY is broken if and only if $\langle F\rangle \neq 0$, and ψ is said to be a Goldstone fermion, or goldstino; note that it is not the SUSY partner of a Goldstone boson. There is no Goldstone boson, as SUSY, being a fermionic symmetry, breaks an assumption of Goldstone's theorem.

- Recalling that the contribution to the scalar potential is

$$V_F = K_{i\bar{j}} F^i F^{\bar{j}*}$$

  we see that SUSY is broken if $\langle V_F \rangle > 0$, corresponding to a positive vacuum energy at the classical level, just as at the quantum level. Equivalently, a SUSY-invariant vacuum is one where $\partial W/\partial\varphi = 0$.

Note. Reformulating the above in mathematical language. As we've seen above, the scalar fields are a map from spacetime to a Kahler manifold; we can think of the φ_i as local coordinates on this manifold. The vacuum equations $\partial W/\partial\varphi_i = 0$ are a set of holomorphic algebraic equations and hence define a subvariety of this manifold. If the subvariety is trivial, SUSY is necessarily spontaneously broken. If there are a continuum of solutions, we say there is a vacuum moduli space.

Example. The Wess–Zumino model. We take the canonical Kahler potential and superpotential

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3.$$

The condition for a vacuum solution is

$$\frac{\partial W}{\partial \varphi} = m\varphi + \lambda\varphi^2 = 0$$

which gives the two discrete supersymmetric vacua

$$\varphi = 0, \quad \varphi = -\frac{m}{\lambda}.$$

Example. Consider a theory with three chiral superfields, the canonical Kahler potential, and

$$W = \Phi_1\Phi_2\Phi_3.$$

The vacuum equations are $\phi_1\phi_2 = \phi_1\phi_3 = \phi_2\phi_3 = 0$, which defines a subvariety; note that the moduli space is not a manifold in this case.

Example. The O'Raifertaigh model. We consider three chiral superfields Φ_1, Φ_2, Φ_3 with

$$K = \Phi_i^\dagger \Phi_i, \quad W = g\Phi_1(\Phi_3^2 - m^2) + M\Phi_2\Phi_3, \quad M \gg m.$$

Using the equations of motion for the auxiliary field F,

$$-F^{1*} = \frac{\partial W}{\partial \varphi_1} = g(\varphi_3^2 - m^2), \quad -F^{2*} = \frac{\partial W}{\partial \varphi_2} = M\varphi_3, \quad -F^{3*} = \frac{\partial W}{\partial \varphi_3} = 2g\varphi_1\varphi_3 + M\varphi_2.$$

Since we cannot have $F_i^* = 0$ for all i simultaneously, this superpotential necessarily breaks SUSY. The scalar potential is

$$V_F(\varphi_i) = F^{1*}F^1 + F^{2*}F^2 + F^{3*}F^3 = g^2|\varphi_3^2 - m^2|^2 + M^2|\varphi_3|^2 + |2g\varphi_1\varphi_3 + M\varphi_2|^2.$$

Hence, since M is large, the potential is minimized at

$$\langle \varphi_2 \rangle = \langle \varphi_3 \rangle = 0, \quad \langle \varphi_1 \rangle \text{ arbitrary}, \quad \langle V \rangle = g^2 m^4 > 0.$$

The only nonzero F field is thus $\langle F_1\rangle \neq 0$. For simplicity, we take $\langle\varphi_1\rangle = 0$. The fermion mass terms are then

$$\left\langle \frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j}\right\rangle \psi^i\psi^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix} \psi^i\psi^j.$$

Hence we have two fermions with mass M and one massless fermion ψ_1, which is the Goldstino corresponding to the nonzero vev of F_1. The quadratic terms in the scalar potential expanded about the vev are

$$V_F(\varphi_i) \supset -m^2 g^2(\varphi_3^2 + \varphi_3^{*2}) + M^2|\varphi_3|^2 + M^2|\varphi_2|^2.$$

Hence the φ_1 field is massless, since it corresponds to a flat direction in the scalar potential, and the φ_2 has mass M. For the φ_3, expand $\varphi_3 = a + bi$ to find

$$m_a^2 = M^2 - 2g^2m^2, \quad m_b^2 = M^2 + 2g^2m^2.$$

We define the supertrace as the trace with an extra minus sign for bosons,

$$\mathrm{STr}(M^2) \equiv \sum_j (-1)^{2j+1}(2j+1)m_j^2 = 0.$$

This result is generic for tree-level SUSY breaking.

Note. We can show that the supertrace vanishes at tree level for arbitrarily many chiral superfields. First note that the fermion mass matrix is

$$(M_F)^{ij} = \langle \partial^i\partial^j W\rangle, \quad \mathrm{tr}\, M_F^\dagger M_F = \langle \partial^i\partial^j W\rangle K_{i\bar{i}}K_{j\bar{j}}\langle \partial^{\bar{i}}\partial^{\bar{j}} W^*\rangle.$$

Now, the scalar potential is

$$V = K_{i\bar{j}}\langle \partial^i W\rangle\langle\partial^{\bar{j}}W^*\rangle$$

which means the scalar mass terms take the form

$$\mathcal{L} \supset -\left(\varphi_{\bar{j}}^*\varphi_i\, \partial^i\partial^{\bar{j}}V + \frac{1}{2}\varphi_i\varphi_j\, \partial^i\partial^j V + \frac{1}{2}\varphi_{\bar{i}}^*\varphi_{\bar{j}}^*\, \partial^{\bar{i}}\partial^{\bar{j}}V\right) = -\frac{1}{2}\boldsymbol{\varphi}^\dagger M_B^2\boldsymbol{\varphi}$$

where we consider φ_i and $\varphi_{\bar{i}}^*$ as independent real fields, and

$$\boldsymbol{\varphi} = \begin{pmatrix}\varphi \\ \varphi^*\end{pmatrix}, \quad M_B^2 = \begin{pmatrix} \partial\bar{\partial}V & \bar{\partial}\bar{\partial}V \\ \partial\partial V & \partial\bar{\partial}V\end{pmatrix}.$$

Hence we have

$$\mathrm{tr}\, M_B^2 = 2\partial\bar{\partial}V = 2K_{i\bar{i}}\partial^i\partial^{\bar{i}}\left(K_{j\bar{j}}\langle\partial^j W\rangle\langle\partial^{\bar{j}}W^*\rangle\right) = 2\langle\partial^i\partial^j W\rangle K_{i\bar{i}}K_{j\bar{j}}\langle\partial^{\bar{i}}\partial^{\bar{j}}W^*\rangle = 2\,\mathrm{tr}\, M_F^\dagger M_F.$$

Since each fermionic field contains two degrees of freedom, $\mathrm{Str}(M^2) = 0$ as desired.

Note. We have shown that W is not renormalized to all orders in perturbation theory; hence if SUSY is unbroken at tree level, it is unbroken in perturbation theory. Moreover, if SUSY is broken at tree level, the supertrace of M^2 vanishes, implying that the superpartners cannot be too much heavier. Since this appears to be experimentally ruled out, SUSY must be broken nonperturbatively.

Example. For a vector superfield $V = (\lambda, A_\mu, D)$ in Wess–Zumino gauge, we must have $\langle \lambda \rangle = \langle A_\mu \rangle = 0$ by Lorentz invariance. However, D can acquire a vev, and since

$$\delta\lambda \propto \epsilon D$$

we see that SUSY can be broken when D acquires a vev, where λ is the Goldstino. This is called D-term SUSY breaking, in contrast with F-term SUSY breaking above. Since the contribution to the scalar potential is proportional to $\langle D \rangle^2$, SUSY is broken if $\langle V_D \rangle > 0$.

In the very simplest case, we consider a single chiral superfield with $U(1)$ charge q and trivial superpotential, where $q > 0$ and $\xi \geq 0$. The scalar potential is

$$V(\varphi) = \frac{1}{2}\left(\frac{\xi}{2} + q|\varphi|^2\right)^2, \quad \langle \varphi \rangle = 0, \quad \langle D \rangle = -\frac{\xi}{2}$$

which means that SUSY is broken when $\xi > 0$. Since $\langle \varphi \rangle = 0$, the $U(1)$ symmetry is unbroken, so the λ and V_μ remain massless. Since the superpotential is trivial, the ψ remains massless. Finally, using the scalar potential, we see

$$V(\varphi) \supset \frac{\xi}{2} q|\varphi|^2, \quad m_\varphi^2 = q\xi/2.$$

On the other hand, if $q > 0$ and $\xi < 0$, then we have

$$|\langle \varphi \rangle|^2 = -\frac{\xi}{2q}, \quad \langle D \rangle = 0$$

which indicates that SUSY is not broken, but the $U(1)$ symmetry is. Then the λ and V_μ fields acquire mass by the ordinary Higgs effect by interacting with the vev of φ.

Note. In the case of D-term breaking, the supertrace sum rule is slightly modified; it turns out to be proportional to the sum of all $U(1)$ charges. However, this quantity must vanish to ensure anomaly cancellation.

Finally, we briefly discuss SUSY breaking in supergravity.

- In supergravity, there is a new auxiliary field F_g, which can break SUSY by acquiring a vev. Specifically, the F-term is

  $$F \propto DW, \quad D_i W \equiv \partial_i W + (\partial_i K) W$$

  where we have set $M_{\text{pl}} = 1$.

- The scalar potential has a negative gravitational contribution,

  $$V = e^K \left(K^{i\bar{j}} D_i W D_{\bar{j}} W^* - 3|W|^2\right).$$

  This is important because it allows $\langle V \rangle = 0$ even after SUSY breaking, which avoids an unacceptably large cosmological constant. However, this does not solve the cosmological constant problem, because $\langle V \rangle$ is generically large and negative.

- There are "no-scale" supergravity models where the Kahler potential and superpotential are chosen so that $\langle V \rangle = 0$, but these are not regarded as a solution of the cosmological problem either, because there is no reason the form of the potentials should be preserved by quantum corrections.

- In the process of SUSY breaking, the gravitino field, which is the gauge field of $\mathcal{N}=1$ supergravity, 'eats' the goldstino and gains mass. This is called the super Higgs effect, and should not be confused with the supersymmetric extension of the ordinary Higgs effect, where a massless vector superfield eats a chiral superfield to gain mass.

5.2 Particles and Interactions

Next, we discuss the matter content of the MSSM.

- The MSSM has $\mathcal{N}=1$ SUSY with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The matter fields are the same as in the SM, with spinor fields promoted to chiral superfields. Note that some conjugations are necessary, since chiral superfields only contain left-chiral spinors.
- Specifically, we have quarks and squarks,

  $$Q_i = (3, 2, 1/6), \quad u_i^c = (\overline{3}, 1, -2/3), \quad d_i^c = (\overline{3}, 1, -1/3)$$

  including, e.g., the stop squark, as well as leptons and sleptons,

  $$L_i = (1, 2, 1/2), \quad e_i^c = (1, 1, 1),$$

  including, e.g. the selectron sneutrino.
- The Higgs acquires a superpartner, the Higgsino. Since the Higgsino contributes to the $U(1)_Y$ anomaly, a second Higgs field with opposite hypercharge is required to cancel it. We have

  $$H_1 = (1, 2, -1/2), \quad H_2 = (1, 2, 1/2)$$

  where the H_1 field is not present in the SM. Hence the MSSM is a two Higgs doublet model. Another way to see a second Higgs is required is that in the SM, we must use the Higgs conjugate field for some of the Yukawa terms, but we can't do that here since the superpotential is holomorphic.
- The gauge bosons correspond to vector superfields, giving gluons and gluinos, W bosons and winos, and B bosons and binos,

  $$G = (8, 1, 0), \quad W = (1, 3, 0), \quad B = (1, 1, 0).$$

  The neutral winos, binos, and Higgsinos mix to form Majorana fermions called neutralinos, the lightest of which could serve as a dark matter candidate; typically this candidate is mostly bino. The charged winos, binos, and Higgsinos form charginos.

We now consider the interactions in the MSSM.

- As in super QED, we have interactions by the chiral superfield kinetic term. However, the FI term must be zero, as otherwise the scalar potential for squarks and sleptons would yield a vacuum breaking $U(1)_A$ and $SU(3)_C$ symmetry.
- Rescaling the gauge fields, the gauge kinetic terms $f_a = \tau_a$ have $\operatorname{Re}\tau_a = 4\pi/g_a^2$, specifying the gauge couplings.

- The most general renormalizable superpotential is given by

$$W = y_1 Q H_2 u^c + y_2 Q H_1 d^c + y_3 L H_1 e^c + \mu H_1 H_2 + \lambda_1 L L e^c + \lambda_2 L Q d^c + \lambda_3 u^c d^c d^c + \mu' L H_2$$

  where we have suppressed generation indices; properly every coefficient is a matrix in generation space. The first three terms yield standard Higgs Yukawa couplings to matter, while the fourth is a mass term for the two Higgs fields.

- The last four terms break either $U(1)_B$ or $U(1)_L$. They are not allowed phenomenologically, because if the parameter values were natural, then protons would decay in seconds.

- The simplest way to forbid these terms is to impose R-parity,

$$R \equiv (-1)^{3(B-L)+2s} = \begin{cases} +1 & \text{all observed particles,} \\ -1 & \text{superpartners} \end{cases}$$

  where s is the spin. This has the additional benefit that the lightest superpartner (LSP) is stable, and hence can serve as a candidate for cold weakly interacting dark matter. In collider experiments, one can search for LSP pair production by 'missing energy'.

- Note that it would have been completely equivalent to define R to be $(-1)^{3(B-L)}$, because all interaction terms are Lorentz scalars, so the spins of the fields involve must sum to an integer. Our definition of R is just slightly nicer. R-parity can be realized by having a $U(1)_{B-L}$ gauge symmetry spontaneously broken at high energies in an appropriate way.

Note. The imposition of R-parity is a useful general model building tool, outside of SUSY. In a generic extension of the Standard Model with an R-parity-like symmetry (which we call T-parity), every vertex has an even number of new particles, and these must be connected up to yield a contribution to a Standard Model operator, so the leading contributions are at loop level. This gives one more room to avoid bounds, and reduces contributions to the Higgs mass. Theories with T-parity also tend to have a dark matter candidate, namely the lightest T-odd particle (LTOP).

Next, we discuss mechanisms for SUSY breaking.

- As shown above, naive SUSY breaking wouldn't work, because $\mathrm{STr}(M^2)$ vanishes, and the superpartners would be too light. Instead, we introduce a hidden sector which breaks SUSY. The hidden sector may obey the sum rule, but it isn't ruled out because it doesn't interact directly with the MSSM fields; instead it interacts through a messenger sector. Typically, the gauge group is enlarged by another factor G, under which all MSSM fields are singlets.

- One possible SUSY breaking mechanism is gaugino condensation. Here an asymptotically free gauge coupling g becomes large at some energy scale M. If we start with a cutoff Λ, then

$$M = \Lambda \exp(g^{-2}(\Lambda)/\beta)$$

  so it is easily possible to have $M \ll \Lambda$. Here SUSY is broken dynamically and nonperturbatively, so the sum rule doesn't apply. This is analogous to dimensional transmutation in QCD, which explains why $\Lambda_{\mathrm{QCD}} \ll M_{\mathrm{pl}}$.

- Next, we need to specify the messenger sector. For example, the mediating field could simply be the graviton. Then couplings are suppressed by M_{pl}, so by dimensional analysis

$$\Delta m = \frac{M^2}{M_{\text{pl}}}$$

  where Δm describes the size of the mass splittings in the MSSM. Setting $\Delta m \sim 1\,\text{TeV}$ and $M_{\text{pl}} \sim 10^{18}\,\text{GeV}$ gives $M \sim 10^{11}\,\text{GeV}$. This scenario requires a gravitino, which acquires a mass Δm by the super Higgs mechanism.

- Another situation is gauge mediation. Here the messenger fields are charged under both G and the SM gauge group, and the SUSY breaking is transmitted by loops. Then

$$\Delta m \sim \frac{M}{16\pi^2}$$

  which means M must also be around the TeV scale. Then the gravitino mass is on the order of $M^2/M_{\text{pl}} \sim \text{eV}$ so it is the LSP.

- To work phenomenologically, we integrate out the messenger sector and hidden sector to yield a Lagrangian for the MSSM with SUSY breaking terms. Generically, we get all possible "soft SUSY breaking terms", i.e. renormalizable terms that do not reintroduce the hierarchy problem (quadratic sensitivity to Λ^2), such as mass terms for superpartners and additional interactions. This is the source of the many (> 100) parameters in the MSSM.

- Almost all of the MSSM parameter space is ruled out, because SUSY particles could heavily mix in general, and this mixing would be transferred to quarks by loops, causing flavor changing neutral currents.

- Specific high scale models provide relations between the parameters. For example, the constrained MSSM (CMSSM), which may arise from string theory has only three free parameters.

- Extra structure is needed to account for neutrino masses. One may also add an additional singlet Higgs (and its superpartner), which resolves some theoretical tensions. The result is the next-to-minimal extension of the SM, the NMSSM.

Finally, we revisit the hierarchy problem.

- We may split the hierarchy problem into two parts: why $M_{\text{ew}} \ll M_{\text{pl}}$ at tree level, and why this is stable under quantum corrections. These are qualitatively distinct applications of naturalness, and both are challenging.

- Note that SUSY introduces new scalar particles, but they don't create new hierarchy problems because they are superpartners of fermions, which are naturally light.

- As argued before, SUSY cancels the quadratic divergences in the Higgs self-energy. We retain logarithmic divergences of the form $\Delta m \log(M/\Delta m)$, which may naturally give a small result as long as Δm is around the TeV scale, motivating low-scale SUSY. An independent argument for TeV scale SUSY is gauge coupling unification. A third argument comes from the WIMP miracle, which is that TeV scale SUSY can account for dark matter by the LSP.

- A more sophisticated way to understand these cancellations comes from the non-renormalization theorems above. The Higgs mass term comes from the superpotential, and we have shown it is not renormalized.
- In principle, SUSY could also solve the cosmological constant problem. However, it is broken at far too high a scale; we would have $M_\Lambda \sim \Delta m$, while in reality $M_\Lambda \ll \Delta m$. At present, there is no satisfactory solution to this problem.

Note. Gauge coupling unification only makes sense in the context of grand unification. The coupling constants for non-abelian gauge theories are normalized by normalizing the generators T^a so that, e.g. $\mathrm{tr}(T^a T^b) = \delta^{ab}/2$. However, there's no canonical way to normalize the $U(1)$ coupling; by different choices of this normalization one can make gauge coupling unification happen for any theory. The only way to resolve this ambiguity is to determine how $U(1)$ is embedded in the GUT group, which fixes its normalization.

Note. The technical naturalness of the smallness of the fermion masses can be seen nicely by spurions. Let the mass parameter be m. Then the Lagrangian maintains chiral symmetry if m is charged under it, which implies corrections to it must take the form $\delta m = mf(|m|^2)$. In general, whenever a symmetry is restored when a parameter vanishes, it can be maintained for nonzero values of that parameter by promoting it to a spurion charged under that symmetry.