Quantum Field Theory I

Final Examination

Stanford University, Autumn 2022

- This is a take-home final exam, worth 30% of the course grade. Submit your solutions on Gradescope by **4 PM** (Pacific time) on **Thursday, December 15**.
- The exam is graded out of 40 points; the starred question is extra credit.
- No late submissions will be accepted, except for extreme circumstances such as medical emergencies; please email us as early as possible if these arise.
- If you have questions about the problems, email both of us. If we find there is an error or omission in the problems, we will send an announcement to the entire class.
- You are allowed to use the course textbook, by Peskin and Schroeder, and all course materials, including lecture notes, section notes, problem sets, and their solutions.
- You may use Mathematica or other computer programs to perform algebra, but you cannot use external packages and must include a copy of your source code.
- Collaboration with other students, or use of other sources or the internet in general, is prohibited. By submitting this exam, you affirm that you have received no unauthorized aid and engaged in no academic dishonesty.

The exam is comparable in length to a problem set, so we encourage you to start early. Note that using online sources is both against the rules and likely not actually helpful. Legitimate sources such as review papers are usually too sophisticated to understand given background at the level of this course. Unvetted sources such as internet forums won't answer the same questions, are frequently wrong, and differ widely in conventions. All of the problems can be solved by mildly extending calculations you have done in the problem sets earlier in the quarter, and none require any knowledge from outside the course. Good luck!

> Not only God knows, I know, and by the end of the semester, you will know.
> — Sidney Coleman, in a Harvard QFT I lecture

1. Field uncertainties. (10 points)

Quantum field theories have states with definite numbers of particles, in analogy with the number states $|n\rangle$ of a quantum harmonic oscillator. However, there also exist states with definite field profiles, in analogy with the position states $|x\rangle$. A general state is a superposition of such field eigenstates, and thus can have an uncertain field value.

a) Consider a free real scalar field ϕ of mass m in the vacuum state. The variance of the field value at $\mathbf{x}$ is

$$\sigma^2_{\phi(\mathbf{x})} = \langle 0|\phi(\mathbf{x})^2|0\rangle - \left(\langle 0|\phi(\mathbf{x})|0\rangle\right)^2. \tag{1}$$

Show that this quantity is infinite.

b) There's nothing wrong with the result in part (a), because in practice we can never measure the field value at a point. We only measure "smeared" field values averaged over some length scale a, such as

$$\phi_a(\mathbf{x}) = \frac{1}{a^3\pi^{3/2}} \int d^3y\, \phi(\mathbf{y}) e^{-|\mathbf{x}-\mathbf{y}|^2/a^2}. \tag{2}$$

Write $\sigma^2_{\phi_a(\mathbf{x})}$ in the form of an integral over a single variable.

c) The standard deviation of the smeared field is finite. Find expressions for $\sigma^2_{\phi_a(\mathbf{x})}$ in the limits $a \ll 1/m$ and $a \gg 1/m$. (Hint: in each regime, your answer should have the form $\alpha\, a^\beta$ for some values of α and β. If you're having trouble doing the final integral in one of these cases, you can start from the result $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.)

d) Consider a cubical copper microwave cavity of volume $V = 1\,\mathrm{m}^3$. The electric field in the cavity does not have a definite value, due to the quantization of the electromagnetic field. Find a rough, order of magnitude estimate for the standard deviation of the average electric field in the cavity in its ground state, and evaluate it in volts per meter. (Hint: you will need to go from natural units back to SI units using $\hbar = 10^{-34}\,\mathrm{J\,s}$, $c = 3 \times 10^8\,\mathrm{m/s}$, and $e = 1.6 \times 10^{-19}\,\mathrm{C}$.)

2. Forces on external currents. (5 points)

In problem set 7, you considered the "improved" electromagnetic stress-energy tensor,

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{3}$$

and showed that it was conserved, $\partial_\mu \hat{T}^{\mu\nu} = 0$, when no current was present.

a) When there is a nonzero current $J^\mu = (\rho, \mathbf{J})$, the stress-energy tensor defined in (3) has divergence $\partial_\mu \hat{T}^{\mu\nu} = -K^\nu$. Find K^ν.

b) The vector $K^\mu = (K^0, \mathbf{K})$ is the four-momentum delivered to the current by the field, per unit space and unit time. Explicitly write K^0 and $\mathbf{K}$ in terms of ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$.

3. Pseudoscalar and vector decays. (10 points)

In this problem we will consider decays of particles to pairs of electrons. As usual, the electron is described by a Dirac field Ψ of mass m_e. All decay rates should be computed in the rest frame of the decaying particle, and you can reuse results from problem sets.

a) Consider a pseudoscalar particle, described by a real scalar field ϕ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = ig\phi\, \bar{\Psi}\gamma^5\Psi. \tag{4}$$

Then find the decay rate for $\phi \to e^+e^-$ to leading nontrivial order in g.

b) Now consider a vector particle, described by a massive vector field A_μ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = g\, \bar{\Psi}\gamma^\mu A_\mu \Psi. \tag{5}$$

Then find the decay rate for $A_\mu \to e^+e^-$ to leading nontrivial order in g. (Hint: the Feynman rule for an incoming massive vector is $\epsilon_\mu(p)$, where ϵ_μ is its polarization. The polarization satisfies $\epsilon \cdot \epsilon = -1$ and $p \cdot \epsilon = 0$, since $\partial_\mu A^\mu = 0$. By rotational symmetry, the decay rate doesn't depend on the vector's polarization, so you can compute it by either averaging over possible polarizations using

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \tag{6}$$

or by picking any specific polarization.)

4. Higgs boson production at a muon collider. (15 points)

The high energy physics community is currently debating how to best study the Higgs boson with future particle accelerators. One idea is to build a machine that collides muons and anti-muons at very high energies and produces Higgs bosons in their annihilation.

You should treat the muon, Higgs, and photon in the same way as in problem set 8, and neglect any other particles or interactions. That is, consider the free Lagrangian

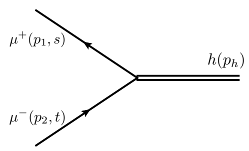
$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2\, h^2 + \bar{\Psi}(i\partial\!\!\!/ - m_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7}$$

with interactions

$$\mathcal{L}_{\text{int}} = -\frac{m_\mu}{v}\, h\bar{\Psi}\Psi - e\bar{\Psi}A\!\!\!/\Psi. \tag{8}$$

We first consider the case where a muon and anti-muon annihilate to produce a Higgs boson, $\mu^+(p_1)\mu^-(p_2) \to h(p_h)$.

a) Compute the scattering amplitude $\mathcal{M}_{p_1\, p_2 \to p_h}$ for this process at lowest nontrivial order in perturbation theory, by explicitly using Wick's theorem. Then show that you get the same answer if you simply apply the Feynman rules, where the vertex is



$$= -\frac{im_\mu}{v}. \tag{9}$$

The scattering cross section of particles a and b with masses m_a and m_b to produce N final state particles is

$$\sigma_{p_a\,p_b\to q_1\,\ldots q_N} = \frac{1}{2\sqrt{(2p_a p_b)^2 - 4m_a^2 m_b^2}} \int \prod_{i=1}^{N} \left(\frac{d^4 q_i}{(2\pi)^4} (2\pi)\delta(q_i^2 - m_i^2)\theta(q_i^0 - m_i) \right) \\ \times (2\pi)^4 \delta^{(4)}(p_a + p_b - q_1 - \cdots - q_N) |\bar{\mathcal{M}}_{p_a\,p_b\to q_1\,\ldots q_N}|^2 \quad (10)$$

where $|\bar{\mathcal{M}}_{p_a\,p_b\to q_1\,\ldots q_N}|^2$ is the modulus squared of the scattering amplitude, summed over final state spins and averaged over initial state spins.

b) Compute $|\bar{\mathcal{M}}_{p_1\,p_2\to p_h}|^2$ and write it in terms of the constants defined in (7) and (8).

c) Using this result and (10), compute the total cross section for this process. Your result should contain distributions – what is their interpretation?

It is also possible to produce additional particles in the final state. For the rest of this problem, we consider the process $\mu^+(p_1)\,\mu^-(p_2) \to \gamma(p_3)\,h(p_h)$ at leading nontrivial order in perturbation theory.

d) Draw all the relevant Feynman diagrams and use the Feynman rules to write down the corresponding amplitude $\mathcal{M}_{p_1 p_2\to p_3 p_h}$.

e) Since a muon collider would operate at a center-of-mass energy much higher than the muon mass, we can treat m_μ as a small quantity. Compute $|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2$ to leading nontrivial order in m_μ, and write it in terms of the Mandelstam invariants

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_h)^2, \qquad u = (p_1 - p_3)^2 \quad (11)$$

and the constants defined in (7) and (8). (Hint: your expression will contain a sum over photon polarizations of the form

$$\sum_\lambda \epsilon^\mu(p_3, \lambda)\epsilon^{*\nu}(p_3, \lambda) \quad (12)$$

which you can simply replace with $-\eta^{\mu\nu}$.)

f) Using this result and (10), compute the differential cross section $d\sigma/d\cos\theta$ for this process in the center-of-mass frame, where θ is the angle between the three-momenta of the muon and the photon. You may again work to leading nontrivial order in m_μ. Express your result in terms of s, θ, and the constants defined in (7) and (8).

5. ⋆ Kaluza–Klein theory. (5 points)

Some theories of physics beyond the Standard Model involve compactified extra spacetime dimensions. In this problem, you will see why this generically gives rise to many new particles, and why the extra dimensions are hard to detect when they are very small.

Consider a massless real scalar field on a five-dimensional spacetime, where the extra dimension is compactified on a circle of radius R. That is, points are labeled by (x, w) where $x = (t, x, y, z)$ is the usual four-dimensional spacetime coordinate, and $w \in [0, 2\pi R]$, with the points $w = 0$ and $w = 2\pi R$ identified. The metric on this spacetime is still mostly negative, so $\eta^{00} = 1$ with the other diagonal elements negative. The action is

$$S = \int d^4x \int_0^{2\pi R} dw\, \frac{1}{2}(\partial_M \phi(x, w))(\partial^M \phi(x, w)) \quad (13)$$

where the index M ranges from 0 to 4.

a) Using Fourier series, the w-dependence of the field can be written as

$$\phi(x,w) = \sum_n \phi^{(n)}(x)\, e^{ik_n w} \tag{14}$$

where the sum is over integer n. What are the k_n, and how is $\phi^{(n)}$ related to $\phi^{(-n)}$?

b) Plug this decomposition into the action and perform the w integral to yield an ordinary four-dimensional action, written in terms of the fields $\phi^{(n)}(x)$ for $n \geq 0$. What are the physical masses m_n of these fields? (Hint: you should rescale the fields to give the kinetic terms the usual normalizations.)

Now consider the case of a massless vector field, which can be decomposed as

$$A_M(x,w) = \sum_n A_M^{(n)}(x) e^{ik_n w}. \tag{15}$$

The action is the five-dimensional analogue of the usual electromagnetic action,

$$S = -\frac{1}{4}\int d^4x \int_0^{2\pi R} dw\, F_{MN}F^{MN} \tag{16}$$

and we have the five-dimensional analogue of the usual gauge symmetry, $A_M \to A_M + \partial_M \alpha$.

c) Show that we can use a gauge transformation to set $A_4^{(n)} = 0$ when $n \neq 0$, and explain why this is not possible when $n = 0$.

d) Using this gauge, find all the resulting four-dimensional fields and their masses.