Quantum Field Theory I
Final Examination

Stanford University, Autumn 2022

- This is a take-home final exam, worth 30% of the course grade. Submit your solutions on Gradescope by **4 PM** (Pacific time) on **Thursday, December 15**.

- The exam is graded out of 40 points; the starred question is extra credit.

- No late submissions will be accepted, except for extreme circumstances such as medical emergencies; please email us as early as possible if these arise.

- If you have questions about the problems, email both of us. If we find there is an error or omission in the problems, we will send an announcement to the entire class.

- You are allowed to use the course textbook, by Peskin and Schroeder, and all course materials, including lecture notes, section notes, problem sets, and their solutions.

- You may use Mathematica or other computer programs to perform algebra, but you cannot use external packages and must include a copy of your source code.

- Collaboration with other students, or use of other sources or the internet in general, is prohibited. By submitting this exam, you affirm that you have received no unauthorized aid and engaged in no academic dishonesty.

The exam is comparable in length to a problem set, so we encourage you to start early. Note that using online sources is both against the rules and likely not actually helpful. Legitimate sources such as review papers are usually too sophisticated to understand given background at the level of this course. Unvetted sources such as internet forums won't answer the same questions, are frequently wrong, and differ widely in conventions. All of the problems can be solved by mildly extending calculations you have done in the problem sets earlier in the quarter, and none require any knowledge from outside the course. Good luck!

> Not only God knows, I know, and by the end of the semester, you will know.
> — Sidney Coleman, in a Harvard QFT I lecture

1. Field uncertainties. (10 points)

Quantum field theories have states with definite numbers of particles, in analogy with the number states $|n\rangle$ of a quantum harmonic oscillator. However, there also exist states with definite field profiles, in analogy with the position states $|x\rangle$. A general state is a superposition of such field eigenstates, and thus can have an uncertain field value.

a) Consider a free real scalar field ϕ of mass m in the vacuum state. The variance of the field value at $\mathbf{x}$ is

$$\sigma^2_{\phi(\mathbf{x})} = \langle 0|\phi(\mathbf{x})^2|0\rangle - \left(\langle 0|\phi(\mathbf{x})|0\rangle\right)^2. \tag{1}$$

Show that this quantity is infinite.

Solution: The expectation value is
$$\langle 0|\phi(\mathbf{x})|0\rangle = 0 \tag{S1}$$

while the expectation value of the square is

$$\langle 0|\phi(\mathbf{x})^2|0\rangle = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}}$$
$$\times \langle 0|\left(a(\mathbf{p})e^{i\mathbf{x}\cdot\mathbf{p}} + a^\dagger(\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{p}}\right)\left(a(\mathbf{q})e^{i\mathbf{x}\cdot\mathbf{q}} + a^\dagger(\mathbf{q})e^{-i\mathbf{x}\cdot\mathbf{q}}\right) \tag{S2}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} \langle 0|a(\mathbf{p})a^\dagger(\mathbf{q})e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})}|0\rangle \tag{S3}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} \tag{S4}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2E(\mathbf{p})} \tag{S5}$$

$$= \frac{1}{2\pi^2} \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2 + m^2}} \tag{S6}$$

which is infinite.

b) There's nothing wrong with the result in part (a), because in practice we can never measure the field value at a point. We only measure "smeared" field values averaged over some length scale a, such as

$$\phi_a(\mathbf{x}) = \frac{1}{a^3\pi^{3/2}} \int d^3y\, \phi(\mathbf{y})e^{-|\mathbf{x}-\mathbf{y}|^2/a^2}. \tag{2}$$

Write $\sigma^2_{\phi_a(\mathbf{x})}$ in the form of an integral over a single variable.

Solution: The expectation value still vanishes, but now we have

$$\langle 0|\phi_a(\mathbf{x})^2|0\rangle = \frac{1}{a^6\pi^3} \int d\mathbf{y}d\mathbf{z}\, e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}} e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}} \langle 0|\phi(\mathbf{y})\phi(\mathbf{z})|0\rangle \tag{S7}$$

$$= \frac{1}{a^6\pi^3} \int d\mathbf{y}d\mathbf{z} \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2E(\vec{p})} e^{i\mathbf{p}(\mathbf{y}-\mathbf{z})} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}} e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}}. \tag{S8}$$

Next, we perform the shifts $\mathbf{z} \to \mathbf{z} + \mathbf{x}$ and $\mathbf{y} \to \mathbf{y} + \mathbf{x}$. Then the integral

$$\int d\mathbf{y}\, e^{i\mathbf{p}(\mathbf{y})}e^{-\frac{|\mathbf{y}|^2}{a^2}} = \int d\mathbf{y}\, e^{i|\mathbf{p}||\mathbf{y}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} \tag{S9}$$

$$= (2\pi)\int_0^\infty d|\mathbf{y}||\mathbf{y}|^2 \int_{-1}^1 d\cos\theta|\mathbf{y}|e^{i|\mathbf{p}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} \tag{S10}$$

$$= \frac{(4\pi)}{|\mathbf{p}|} \int_0^\infty d|\mathbf{y}||\mathbf{y}|\sin(|\mathbf{y}||\mathbf{p}|)e^{-\frac{|\mathbf{y}|^2}{a^2}} \tag{S11}$$

$$= a^3\pi^{3/2}e^{-\frac{|\mathbf{p}|^2a^2}{4}}. \tag{S12}$$

Inserting this above, we find

$$\langle 0 | \phi_a(\mathbf{x})^2 | 0 \rangle = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2E(\vec{p})} e^{-\frac{|\mathbf{p}|^2 a^2}{2}}. \tag{S13}$$

Simplifying, we thus conclude that

$$\sigma_{\phi_a(\mathbf{x})}^2 = \langle 0 | \phi_a(\mathbf{x})^2 | 0 \rangle = \frac{1}{4\pi^2} \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2 + m^2}} e^{-\frac{|\mathbf{p}|^2 a^2}{2}}. \tag{S14}$$

c) The standard deviation of the smeared field is finite. Find expressions for $\sigma_{\phi_a(\mathbf{x})}^2$ in the limits $a \ll 1/m$ and $a \gg 1/m$. (Hint: in each regime, your answer should have the form $\alpha \, a^\beta$ for some values of α and β. If you're having trouble doing the final integral in one of these cases, you can start from the result $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$.)

Solution: Changing to the variable $t = a|\mathbf{p}|$, we have

$$\sigma_{\phi_a(\mathbf{x})}^2 = \frac{1}{4\pi^2 a^2} \int_0^\infty dt \frac{t^2}{\sqrt{t^2 + a^2 m^2}} e^{-\frac{t^2}{2}}. \tag{S15}$$

Expanding around $ma \ll 1$, we find

$$\lim_{ma \to 0} \sigma_{\phi_a(\mathbf{x})}^2 = \frac{1}{4\pi^2 a^2} \int_0^\infty dt \, t e^{-\frac{t^2}{2}} = \frac{1}{4\pi^2 a^2}. \tag{S16}$$

Expanding around $ma \gg 1$, we find

$$\lim_{1/(ma) \to 0} \sigma_{\phi_a(\mathbf{x})}^2 = \frac{1}{4\pi^2 a^3 m} \int_0^\infty dt \, t^2 e^{-\frac{t^2}{2}} = \frac{1}{4\pi^{3/2} a^3 m \sqrt{2}}. \tag{S17}$$

d) Consider a cubical copper microwave cavity of volume $V = 1 \, \text{m}^3$. The electric field in the cavity does not have a definite value, due to the quantization of the electromagnetic field. Find a rough, order of magnitude estimate for the standard deviation of the average electric field in the cavity in its ground state, and evaluate it in volts per meter. (Hint: you will need to go from natural units back to SI units using $\hbar = 10^{-34} \, \text{J s}$, $c = 3 \times 10^8 \, \text{m/s}$, and $e = 1.6 \times 10^{-19} \, \text{C}$.)

Solution: In part (c), we showed that for a massless vector field, the standard deviation of the field average over a region of size a is $\sigma \sim 1/a$. The same logic applies to a massless vector field A_μ. The electric field in the cavity is related to A_μ by one spatial or temporal derivative, which by dimensional analysis also gives a factor of $1/a$. Thus, we expect $\sigma_E \sim 1/a^2 \sim 1/V^{2/3}$. (We could also have jumped directly to this result. We already know $[E] = 2$ from problem set 1, and part (c) tells us the answer is neither zero nor infinity, so by dimensional analysis, $1/a^2$ is the only possible answer.)

Going from natural units back to SI units, we have

$$\sigma_E \sim \frac{\hbar c}{e a^2} \sim 10^{-7} \, \text{V/m}. \tag{S18}$$

This might be surprisingly large if you haven't seen it before, but it lines up with typical "quantum noise" scales in precision electronics. Any answer within a few orders of magnitude of this is acceptable.

2. Forces on external currents. (5 points)

In problem set 7, you considered the "improved" electromagnetic stress-energy tensor,

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} + \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{3}$$

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and showed that it was conserved, $\partial_\mu \hat{T}^{\mu\nu} = 0$, when no current was present.

a) When there is a nonzero current $J^\mu = (\rho, \mathbf{J})$, the stress-energy tensor defined in (3) has divergence $\partial_\mu \hat{T}^{\mu\nu} = -K^\nu$. Find K^ν.

Solution: The only difference is that the equation of motion is now $\partial_\mu F^{\mu\nu} = J^\nu$. We already know that when $\partial_\mu F^{\mu\nu} = 0$, the stress-energy is conserved, so to compute K^ν we only need to consider what terms appear which are proportional to J^μ. This can only occur when the derivative ∂_μ hits the factor of $F^{\mu\rho}$ in the first term, so

$$\partial_\mu \hat{T}^{\mu\nu} = -\eta_{\rho\sigma}(\partial_\mu F^{\mu\rho})F^{\nu\sigma} = -\eta_{\rho\sigma}J^\rho F^{\nu\sigma}. \tag{S19}$$

Thus, the answer is $K^\nu = J_\sigma F^{\nu\sigma}$.

You might be wondering why we are allowed to use the original expression for the stress-energy tensor, as rerunning the Noether/Belinfante procedure will give additional terms when J^μ is nonzero. The point is that we're trying to find the change in the energy-momentum of the electromagnetic field alone, so we shouldn't include those extra terms, which represent something like potential energy-momentum.

b) The vector $K^\mu = (K^0, \mathbf{K})$ is the four-momentum delivered to the current by the field, per unit space and unit time. Explicitly write K^0 and $\mathbf{K}$ in terms of ρ, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$.

Solution: Expanding the components out, we have

$$K^i = J_\sigma F^{i\sigma} = J_0 F^{i0} + J_j F^{ij} = J_0 E^i - J_j \epsilon^{ijk} B^k. \tag{S20}$$

Noting that $J_0 = J^0 = \rho$ and $J_i = -J^i$, we conclude

$$\mathbf{K} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{S21}$$

which of course is just the continuous version of the Lorentz force law. Similarly, we have

$$K^0 = J_i F^{0i} = (-J^i)(-F^{i0}) = \mathbf{J} \cdot \mathbf{E} \tag{S22}$$

which is the rate at which the electric field does work on the charges.

3. Pseudoscalar and vector decays. (10 points)

In this problem we will consider decays of particles to pairs of electrons. As usual, the electron is described by a Dirac field Ψ of mass m_e. All decay rates should be computed in the rest frame of the decaying particle, and you can reuse results from problem sets.

a) Consider a pseudoscalar particle, described by a real scalar field ϕ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = ig\phi\,\bar{\Psi}\gamma^5\Psi. \tag{4}$$

Then find the decay rate for $\phi \to e^+ e^-$ to leading nontrivial order in g.

Solution: As usual, the Feynman rule comes from $-iH_{\text{int}} = iL_{\text{int}}$, so the vertex factor is $-g\gamma^5$. (This looks unusual, but it's correct: there had to be an i in L_{int} to ensure the action is real when g is.) Then

$$\mathcal{M} = -g\bar{u}_r(p_1)\gamma^5 v_s(p_2). \tag{S23}$$

Squaring and summing over final spin states gives

$$|\bar{\mathcal{M}}|^2 = g^2 \sum_{r,s} \bar{u}_r(p_1)\gamma^5 v_s(p_2)v_s^\dagger(p_2)(\gamma^5)^\dagger(\gamma^0)^\dagger u_r(p_1) \tag{S24}$$

$$= -g^2 \operatorname{tr}((\slashed{p}_1 + m_e)\gamma^5(\slashed{p}_2 - m_e)\gamma^5) \tag{S25}$$

$$= g^2 \operatorname{tr}((\slashed{p}_1 + m_2)(\slashed{p}_2 + m_e)\gamma^5\gamma^5) \tag{S26}$$

$$= 4g^2(p_1 \cdot p_2 + m_e^2) \tag{S27}$$

$$= 2g^2 M^2. \tag{S28}$$

Plugging this into the decay rate result derived in problem set 5 gives

$$\Gamma = \frac{g^2}{8\pi}\sqrt{M^2 - 4m_e^2}. \tag{S29}$$

b) Now consider a vector particle, described by a massive vector field A_μ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = g\,\bar{\Psi}\gamma^\mu A_\mu \Psi. \tag{5}$$

Then find the decay rate for $A_\mu \to e^+e^-$ to leading nontrivial order in g. (Hint: the Feynman rule for an incoming massive vector is $\epsilon_\mu(p)$, where ϵ_μ is its polarization. The polarization satisfies $\epsilon \cdot \epsilon = -1$ and $p \cdot \epsilon = 0$, since $\partial_\mu A^\mu = 0$. By rotational symmetry, the decay rate doesn't depend on the vector's polarization, so you can compute it by either averaging over possible polarizations using

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2} \tag{6}$$

or by picking any specific polarization.)

Solution: The Feynman rule for the vertex is $ig\gamma^\mu$, as in ordinary QED, so

$$\mathcal{M} = ig\epsilon_\mu^\lambda \bar{u}_r(p_1)\gamma^\mu v_s(p_2). \tag{S30}$$

Summing over final spin states and averaging over the initial three spin states, we have

$$|\bar{\mathcal{M}}|^2 = \frac{g^2}{3}\left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right)\sum_{rs} \bar{u}_r(p_1)\gamma^\mu v_s(p_2)\bar{v}_2(p_2)\gamma^\nu u_r(p_1). \tag{S31}$$

Note that the $p_\mu p_\nu$ terms won't contribute anything, as by the Dirac equation,

$$\bar{u}_r(p_1)\slashed{p}v_s(p_2) = \bar{u}_r(p_1)(\slashed{p}_1 + \slashed{p}_2)v_s(p_2) = \bar{u}_r(p_1)(m_e - m_e)v_s(p_2) = 0. \tag{S32}$$

Thus, keeping just the metric term and rewriting the spinor sum as a trace,

$$|\bar{\mathcal{M}}|^2 = -\frac{g^2}{3}\eta_{\mu\nu}\,\text{tr}((\slashed{p}_1 + m_e)\gamma^\mu(\slashed{p}_2 - m_e)\gamma^\nu) \tag{S33}$$

$$= -\frac{4g^2}{3}\eta_{\mu\nu}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (p_1 \cdot p_2 - m_e^2)\eta^{\mu\nu}) \tag{S34}$$

$$= \frac{8g^2}{3}(p_1 \cdot p_2 + 2m_e^2) \tag{S35}$$

$$= \frac{4g^2 M^2}{3}\left(1 + \frac{2m_e^2}{M^2}\right). \tag{S36}$$

Plugging this into our decay rate result gives

$$\Gamma = \frac{g^2}{12\pi}\left(1 + \frac{2m_e^2}{M^2}\right)\sqrt{M^2 - 4m_e^2}. \tag{S37}$$

4. Higgs boson production at a muon collider. (15 points)

The high energy physics community is currently debating how to best study the Higgs boson with future particle accelerators. One idea is to build a machine that collides muons and anti-muons at very high energies and produces Higgs bosons in their annihilation.

You should treat the muon, Higgs, and photon in the same way as in problem set 8, and neglect any other particles or interactions. That is, consider the free Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2\, h^2 + \bar{\Psi}(i\slashed{\partial} - m_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7}$$

with interactions

$$\mathcal{L}_{\text{int}} = -\frac{m_\mu}{v}\, h\bar{\Psi}\Psi - e\bar{\Psi}\slashed{A}\Psi. \tag{8}$$

We first consider the case where a muon and anti-muon annihilate to produce a Higgs boson, $\mu^+(p_1)\mu^-(p_2) \to h(p_h)$.

a) Compute the scattering amplitude $\mathcal{M}_{p_1 p_2 \to p_h}$ for this process at lowest nontrivial order in perturbation theory, by explicitly using Wick's theorem. Then show that you get the same answer if you simply apply the Feynman rules, where the vertex is
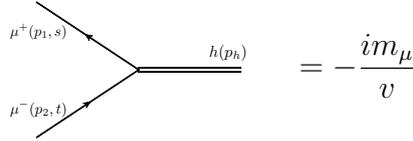
$$= -\frac{im_\mu}{v}. \tag{9}$$

Solution: The initial and final states are

$$|\mu^+(p_1,s), \mu^-(p_2,t)\rangle = \sqrt{2E(\vec{p_1})}\sqrt{2E(\vec{p_2})}b_s^\dagger(p_1)a_t^\dagger(p_2)|0\rangle, \tag{S38}$$

$$|h\rangle = \sqrt{2E(\vec{p_h})}a^\dagger(p_h)|0\rangle. \tag{S39}$$

We can then compute for example the S-matrix element

$$\langle\mu^+(p_1,s), \mu^-(p_2,t)|\int d^4x\, i\mathcal{L}_{int}|h\rangle = \langle\mu^+(p_1,s), \mu^-(p_2,t)|\int d^4x\, i\frac{m_h}{v}h(x)\bar{\Psi}_i(x)\Psi_i(x)|h\rangle$$

$$= \frac{-im_\mu}{v}\sqrt{8E(p_h)E(p_1)E(p_2)}\langle 0|\int d^4x\, b_s(p_1)a_t(p_2)h(x)\bar{\Psi}_i(x)\Psi_i(x)a^\dagger(p_h)|0\rangle. \tag{S40}$$

We find for example for

$$h(x)a^\dagger(p_h) = \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}\left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right]a^\dagger(p_h)$$

$$= a^\dagger(p_h)h(x) + \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}[a(p), a^\dagger(p_h)]e^{-ipx}$$

$$= a^\dagger(p_h)h(x) + \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p_h})e^{-ipx}$$

$$= a^\dagger(p_h)h(x) + \frac{1}{\sqrt{2E(p_h)}}e^{-ip_h x} \tag{S41}$$

In our vacuum expectation value of (S40), the creation operator of the scalar field then acts directly on the vacuum state on the left and annihilates it and only the commutator term remains. Similarly, we find

$$a_t(p_2)\Psi_i(x) = u_i^t(p_2)\frac{e^{ip_2 x}}{\sqrt{2E(p_2)}} - \Psi_i(x)a_t(p_2).$$

$$b_s(p_1)\bar{\Psi}_i(x) = \bar{v}_i^s(p_1)\frac{e^{ip_1 x}}{\sqrt{2E(p_1)}} - \bar{\Psi}_i(x)b_s(p_1). \tag{S42}$$

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As

$$\{b_s^\dagger(p_1), \Psi(x)\} = \{\bar{\Psi}_i(x), a_t^\dagger(p_2)\} = 0, \tag{S43}$$

only the anti-commutator terms survive. We find

$$\langle \mu^+(p_1, s), \mu^-(p_2, t)| \int d^4x i \mathcal{L}_{int}|h\rangle = -\frac{im_\mu}{v} \int d^4x e^{-ix(p_h - p_1 - p_2)} \bar{v}_i^s(p_1)\delta_{ij}u_j^t(p_2)$$

$$= (2\pi)^4\delta^{(4)}(p_h - p_1 - p_2)\bar{v}_i^s(p_1)\left(-\frac{im_\mu}{v}\delta_{ij}\right)u_j^t(p_2). \tag{S44}$$

The scattering matrix elements are defined by

$$S_{fi} = \delta_{fi} + (2\pi)^4\delta(p_f - p_i)\mathcal{M}, \tag{S45}$$

and we associate $u_i^t(p_2)$ with an ingoing muon and $\bar{v}_j^s(p_1)$ with an ingoing anti-muon. We thus conclude

$$\mathcal{M}_{p_1 p_2 \to p_h} = \bar{v}_i^s(p_1)\left(-i\frac{m_\mu}{v}\right)u_i^t(p_2) \tag{S46}$$

which is clearly also what we get from the Feynman rules.

The scattering cross section of particles a and b with masses m_a and m_b to produce N final state particles is

$$\sigma_{p_a p_b \to q_1 \dots q_N} = \frac{1}{2\sqrt{(2p_a p_b)^2 - 4m_a^2 m_b^2}} \int \prod_{i=1}^{N} \left(\frac{d^4q_i}{(2\pi)^4}(2\pi)\delta(q_i^2 - m_i^2)\theta(q_i^0 - m_i)\right)$$

$$\times (2\pi)^4\delta^{(4)}(p_a + p_b - q_1 - \dots - q_N)|\bar{\mathcal{M}}_{p_a p_b \to q_1 \dots q_N}|^2 \tag{10}$$

where $|\bar{\mathcal{M}}_{p_a p_b \to q_1 \dots q_N}|^2$ is the modulus squared of the scattering amplitude, summed over final state spins and averaged over initial state spins.

b) Compute $|\bar{\mathcal{M}}_{p_1 p_2 \to p_h}|^2$ and write it in terms of the constants defined in (7) and (8).

Solution: Using the result of part (a), we have

$$|\bar{\mathcal{M}}_{p_1 p_2 \to p_h}|^2 = \frac{1}{4}\sum_{s,t} \bar{u}_i^t(p_2)\left(i\frac{m_\mu}{v}\right)v_i^s(p_1)\bar{v}_j^s(p_1)\left(-i\frac{m_\mu}{v}\right)u_j^t(p_2) \tag{S47}$$

$$= \frac{m_\mu^2}{4v^2}\text{tr}[(\not{p}_1 - m_\mu)(\not{p}_2 + m_\mu)] \tag{S48}$$

$$= \frac{m_\mu^2}{4v^2}\text{tr}[\not{p}_1\not{p}_2] - 4m_\mu^2 \tag{S49}$$

$$= \frac{m_\mu^2}{2v^2}(m_h^2 - 4m_\mu^2). \tag{S50}$$

The factor of $1/4$ arises due to averaging over 2 spin states per initial state fermion.

c) Using this result and (10), compute the total cross section for this process. Your result should contain distributions – what is their interpretation?

Solution: Introducing $s = (p_1 + p_2)^2$, we find

$$\sigma_{p_1 p_2 \to p_h} = \frac{\pi}{\sqrt{s(s - 4m_\mu^2)}}\int d^4p_h\delta(p_h^2 - m_h^2)\theta(p_h^0 - m_h)\delta^{(4)}(p_1 + p_2 - p_h)|\bar{\mathcal{M}}_{p_1 p_2 \to p_h}|^2 \tag{S51}$$

$$= \frac{\pi}{\sqrt{s(s - 4m_\mu^2)}}\delta(s - m_h^2)\theta(p_1^0 + p_2^0 - m_h)|\bar{\mathcal{M}}_{p_1 p_2 \to p_h}|^2 \tag{S52}$$

$$= \frac{\pi m_\mu^2}{2v^2}\sqrt{1 - \frac{4m_\mu^2}{m_h^2}}\delta(s - m_h^2)\theta(p_1^0 + p_2^0 - m_h). \tag{S53}$$

The distributions forces the center-of-mass energy s to be exactly equal to the Higgs boson mass, which makes sense by energy-momentum conservation. (There is something weird about this result, though: in physics you can never have two things be *exactly* equal to each other, so this would suggest Higgs bosons are never produced this way at all. The resolution is that all unstable particles actually have a slightly indefinite mass (heuristically due to the energy-time uncertainty principle applied to their finite decay time) so the delta function is replaced with a narrow peak.)

It is also possible to produce additional particles in the final state. For the rest of this problem, we consider the process $\mu^+(p_1)\,\mu^-(p_2) \to \gamma(p_3)\,h(p_h)$ at leading nontrivial order in perturbation theory.

d) Draw all the relevant Feynman diagrams and use the Feynman rules to write down the corresponding amplitude $\mathcal{M}_{p_1 p_2 \to p_3 p_h}$.

Solution: There are two relevant diagrams:

$$= \bar{v}_s(p_1)\left(-i\frac{m_\mu}{v}\right)\frac{i(\slashed{p}_1 - \slashed{p}_h + m_\mu)}{(p_1 - p_h)^2 - m_\mu^2}(-ie\gamma^\mu)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S54}$$

$$= \bar{v}_s(p_1)(-ie\gamma^\mu)\frac{i(\slashed{p}_1 - \slashed{p}_3 + m_\mu)}{(p_1 - p_3)^2 - m_\mu^2}\left(-i\frac{m_\mu}{v}\right)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S55}$$

The amplitude is $D_1 + D_2$. All other diagrams are suppressed by more powers of m_μ/v, or more powers of e, or both.

e) Since a muon collider would operate at a center-of-mass energy much higher than the muon mass, we can treat m_μ as a small quantity. Compute $|\mathcal{M}_{p_1 p_2 \to p_3 p_h}|^2$ to leading nontrivial order in m_μ, and write it in terms of the Mandelstam invariants

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_h)^2, \qquad u = (p_1 - p_3)^2 \tag{11}$$

and the constants defined in (7) and (8). (Hint: your expression will contain a sum over photon polarizations of the form

$$\sum_\lambda \epsilon^\mu(p_3,\lambda)\epsilon^{*\nu}(p_3,\lambda) \tag{12}$$

which you can simply replace with $-\eta^{\mu\nu}$.)

Solution: First, we find the leading term in m_μ for our diagrams:

$$D_1 = \frac{-iem_\mu}{vt}\bar{v}_s(p_1)(\slashed{p}_1 - \slashed{p}_h)\gamma^\mu u_t(p_2)\epsilon_\mu(p_3,\lambda), \tag{S56}$$

$$D_2 = \frac{-iem_\mu}{vu}\bar{v}_s(p_1)\gamma^\mu(\slashed{p}_h - \slashed{p}_2)u_t(p_2)\epsilon_\mu(p_3,\lambda). \tag{S57}$$

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Since these quantities already have a factor of m_μ in front, and we only want the lowest order in m_μ, we can set m_μ to zero when evaluating what's left. The Dirac equation for the spinors becomes $\bar{v}_s(p_1)\not{p}_1 = 0 = \not{p}_2 u_t(p_2)$, from which we find

$$D_1 = \frac{iem_\mu}{vt} \bar{v}_s(p_1)\not{p}_h \gamma^\mu u_t(p_2)\epsilon_\mu(p_3, \lambda), \tag{S58}$$

$$D_2 = \frac{-iem_\mu}{vu} \bar{v}_s(p_1)\gamma^\mu \not{p}_h u_t(p_2)\epsilon_\mu(p_3, \lambda). \tag{S59}$$

Next, we compute the terms in the square of the total amplitude,

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_1^\dagger = -\frac{e^2 m_\mu^2}{4v^2 t^2} \mathrm{tr}[\not{p}_1 \not{p}_4 \gamma_\mu \not{p}_2 \gamma^\mu \not{p}_4] = \frac{e^2 m_\mu^2}{2v^2 t^2} \mathrm{tr}[\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_4] = \frac{e^2 m_\mu^2 u}{v^2 t}. \tag{S60}$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_2^\dagger = \frac{e^2 m_\mu^2}{4v^2 tu} \mathrm{tr}[\not{p}_1 \not{p}_4 \gamma_\mu \not{p}_2 \not{p}_4 \gamma^\mu] = \frac{4e^2 m_\mu^2}{v^2 tu}(p_1 p_4)(p_2 p_4) = \frac{e^2 m_\mu^2}{v^2 tu}(m_h^2 - t)(m_h^2 - u). \tag{S61}$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_2 D_2^\dagger = -\frac{e^2 m_\mu^2}{4v^2 u^2} \mathrm{tr}[\not{p}_1 \gamma_\mu \not{p}_4 \not{p}_2 \not{p}_4 \gamma^\mu] = \frac{e^2 m_\mu^2}{2v^2 u^2} \mathrm{tr}[\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_4] = \frac{e^2 m_\mu^2 t}{v^2 u}. \tag{S62}$$

Above, we used the identities

$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p}, \tag{S63}$$

$$\gamma^\mu \not{p}\not{q}\gamma_\mu = 4pq, \tag{S64}$$

$$\mathrm{tr}[\not{p}\not{q}] = 4pq, \tag{S65}$$

$$p_1 p_4 = \frac{m_h^2 - t}{2}, \tag{S66}$$

$$p_2 p_4 = \frac{m_h^2 - u}{2}, \tag{S67}$$

$$m_h^2 = s + t + u, \tag{S68}$$

$$\mathrm{tr}[\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_4] = 4\left[2(p_1 p_4)(p_2 p_4) - (p_1 p_2)p_4^2\right] = 2tu \tag{S69}$$

several of which follow from neglecting m_μ. After some algebra, we find

$$|\bar{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}|^2 = \frac{1}{4}\sum_{s,t,\lambda}\left[D_1 D_1^\dagger + 2\,\mathrm{Re}\left(D_1 D_2^\dagger\right) + D_2 D_2^\dagger\right] = \frac{e^2 m_\mu^2}{v^2 tu}\left(s^2 + m_h^4\right). \tag{S70}$$

f) Using this result and (10), compute the differential cross section $d\sigma/d\cos\theta$ for this process in the center-of-mass frame, where θ is the angle between the three-momenta of the muon and the photon. You may again work to leading nontrivial order in m_μ. Express your result in terms of s, θ, and the constants defined in (7) and (8).

Solution: Starting from (10), we have

$$\sigma_{p_1 p_2 \to p_3 p_h} = \frac{1}{2(2\pi)^2 s}\int d^4 p_3 d^4 p_h \delta(p_3^2)\delta(p_h^2 - m_h^2)\theta(p_3^0)\theta(p_h^0 - m_h)$$
$$\times \delta^{(4)}(p_1 + p_2 - p_3 - p_h)|\bar{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}|^2. \tag{S71}$$

We work in the center-of-mass frame where $s = E_{CM}^2$ and do the p_h integral, giving

$$\sigma_{p_1 p_2 \to p_3 p_h} = \frac{1}{2(2\pi)^2 s}\int d^4 p_3 \delta(p_3^2)\delta(E_{CM}^2 - 2E_{CM}p_3^0 - m_h^2)\theta(p_3^0)\theta(E_{CM} - p_3^0 - m_h)|\bar{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}|^2. \tag{S72}$$

We parametrize the p_3 integral by writing

$$\int d^4 p_3 \delta(p_3^2) = \int_0^\infty dp_3^0 \frac{d|\vec{p}_3|^2}{2}|\vec{p}_3|\int d\cos\theta\, d\phi\, \delta((p_3^0)^2 - |\vec{p}_3|^2) \tag{S73}$$

$$= \pi\int_0^\infty dp_3^0 p_3^0 \int d\cos\theta. \tag{S74}$$

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We can integrate over p_3^0 using the remaining delta function,

$$\delta(E_{CM}^2 - 2E_{CM}p_3^0 - m_h^2) = \frac{1}{2E_{CM}}\delta\left(p_3^0 - \frac{1}{2}(E_{CM} - \frac{m_h^2}{E_{CM}})\right). \tag{S75}$$

Since we want a differential cross section, we simply drop the last integral over $\cos\theta$. Using

$$t = (p_2 - p_3)^2 = -2p_2p_3 = -E_{CM}p_3^0(1 - \cos\theta) \tag{S76}$$
$$u = (p_1 - p_3)^2 = -2p_1p_3 = -E_{CM}p_3^0(1 + \cos\theta) \tag{S77}$$

we find

$$\frac{d\sigma_{p_1\,p_2\to p_3\,p_h}}{d\cos\theta} = \frac{s - m_h^2}{32s^2\pi}|\bar{\mathcal{M}}_{p_1\,p_2\to p_3\,p_h}|^2 = \frac{e^2 m_\mu^2}{8\pi v^2 s^2}\frac{s^2 + m_h^4}{(s - m_h^2)(1 - \cos^2\theta)}. \tag{S78}$$

5. ⋆ Kaluza–Klein theory. (5 points)

Some theories of physics beyond the Standard Model involve compactified extra space-time dimensions. In this problem, you will see why this generically gives rise to many new particles, and why the extra dimensions are hard to detect when they are very small.

Consider a massless real scalar field on a five-dimensional spacetime, where the extra dimension is compactified on a circle of radius R. That is, points are labeled by (x, w) where $x = (t, x, y, z)$ is the usual four-dimensional spacetime coordinate, and $w \in [0, 2\pi R]$, with the points $w = 0$ and $w = 2\pi R$ identified. The metric on this spacetime is still mostly negative, so $\eta^{00} = 1$ with the other diagonal elements negative. The action is

$$S = \int d^4x \int_0^{2\pi R} dw\, \frac{1}{2}(\partial_M\phi(x, w))(\partial^M\phi(x, w)) \tag{13}$$

where the index M ranges from 0 to 4.

a) Using Fourier series, the w-dependence of the field can be written as

$$\phi(x, w) = \sum_n \phi^{(n)}(x)\, e^{ik_n w} \tag{14}$$

where the sum is over integer n. What are the k_n, and how is $\phi^{(n)}$ related to $\phi^{(-n)}$?

Solution: Because the function is periodic in w with period $2\pi R$, we must have $k_n = n/R$. Since the field is real, we must have $(\phi^{(n)})^* = \phi^{(-n)}$.

b) Plug this decomposition into the action and perform the w integral to yield an ordinary four-dimensional action, written in terms of the fields $\phi^{(n)}(x)$ for $n \geq 0$. What are the physical masses m_n of these fields? (Hint: you should rescale the fields to give the kinetic terms the usual normalizations.)

Solution: The result is

$$S = \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw\, \frac{1}{2}\partial_M(\phi^{(n)}e^{inw/R})\partial^M(\phi^{(n')}e^{in'w/R}) \tag{S79}$$

$$= \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw\, \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(n')} - \frac{in}{R}\frac{in'}{R}\phi^{(n)}\phi^{(n')}\right)e^{inw/R}e^{in'w/R} \tag{S80}$$

$$= \int d^4x \sum_{n,n'} \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(n')} - \frac{in}{R}\frac{in'}{R}\phi^{(n)}\phi^{(n')}\right)2\pi R\delta_{n,-n'} \tag{S81}$$

$$= \int d^4x \sum_n \frac{1}{2}\left(\partial_\mu\phi^{(n)}\partial^\mu\phi^{(-n)} - \frac{n^2}{R^2}\phi^{(n)}\phi^{(-n)}\right)2\pi R. \tag{S82}$$

Pairing up terms with positive and negative n, we get

$$S = 2\pi R \int d^4x \, \frac{1}{2} \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n>0} \left(|\partial_\mu \phi^{(n)}|^2 - \frac{n^2}{R^2} |\phi^{(n)}|^2 \right). \tag{S83}$$

This is almost the action for a real scalar field and a bunch of complex scalar fields, but we need to rescale to get the right kinetic terms. Defining $\varphi^{(n)} = \sqrt{2\pi R} \, \phi^{(n)}$, we conclude

$$S = \int d^4x \, \frac{1}{2} \partial_\mu \varphi^{(0)} \partial^\mu \varphi^{(0)} + \sum_{n>0} \left(|\partial_\mu \varphi^{(n)}|^2 - \frac{n^2}{R^2} |\varphi^{(n)}|^2 \right) \tag{S84}$$

from which we see $m_0 = 0$ and $m_n = n/R$. We get an infinite tower of increasingly massive particles, whose masses are very high if R is very small.

Now consider the case of a massless vector field, which can be decomposed as

$$A_M(x, w) = \sum_n A_M^{(n)}(x) e^{ik_n w}. \tag{15}$$

The action is the five-dimensional analogue of the usual electromagnetic action,

$$S = -\frac{1}{4} \int d^4x \int_0^{2\pi R} dw \, F_{MN} F^{MN} \tag{16}$$

and we have the five-dimensional analogue of the usual gauge symmetry, $A_M \to A_M + \partial_M \alpha$.

c) Show that we can use a gauge transformation to set $A_4^{(n)} = 0$ when $n \neq 0$, and explain why this is not possible when $n = 0$.

Solution: We can set $A_4^{(n)} = 0$ for all $n \neq 0$ by taking

$$\alpha = -\sum_{n \neq 0} A_M^{(n)}(x) \frac{e^{ik_n w}}{ik_n}. \tag{S85}$$

This doesn't work for $n = 0$ because it would result in division by zero. Or, in gratuitously fancy language, $A_4^{(0)}$ corresponds to the constant function on the circle, which has zero derivative, yet itself is not the derivative of anything; such a function exists because of the nontrivial cohomology of the circle.

d) Using this gauge, find all the resulting four-dimensional fields and their masses.

Solution: Since we're going to have to rescale the fields later anyway, let's preemptively define

$$A_\mu(x, w) = \sqrt{2\pi R} \sum_n A_\mu^{(n)}(x) e^{ik_n w}, \qquad \phi = \sqrt{2\pi R} \, A_4^{(0)}. \tag{S86}$$

The action becomes

$$S = \int d^4x \int_0^{2\pi R} dw \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F_{\mu 4} F^{\mu 4} \right). \tag{S87}$$

The first term is

$$S_1 = \int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \left(-\frac{1}{4} \sum_{n,n'} e^{i(n+n')w/R} F_{\mu\nu}^{(n)} F^{\mu\nu \, (n')} \right) \tag{S88}$$

$$= -\frac{1}{4} \int d^4x \, F_{\mu\nu}^{(0)} F^{\mu\nu \, (0)} - \frac{1}{2} \sum_{n>0} \int d^4x \, (F_{\mu\nu}^{(n)})^* F^{\mu\nu \, (n)} \tag{S89}$$

which is just the kinetic term for a real vector field $A_\mu^{(0)}$ and a tower of *complex* vector fields $A_\mu^{(n)}$ for $n > 0$. Next, the second term is

$$S = -\frac{1}{2} \int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \left(\partial_\mu \phi - \sum_n \partial_4 (A_\mu^{(n)} e^{inw/R}) \right) \left(-\partial^\mu \phi - \sum_{n'} \partial^4 (A^{\mu\,(n')} e^{in'w/R}) \right) \tag{S90}$$

$$= \int d^4x\, \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} \int d^4x \int_0^{2\pi R} \frac{dw}{2\pi R} \sum_{n,n'} e^{i(n+n')w/R} \frac{nn'}{R^2} A_\mu^{(n)} A^{\mu\,(n')} \tag{S91}$$

$$= \int d^4x\, \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + \sum_{n>0} \int d^4x\, \frac{n^2}{R^2} (A_\mu^{(n)})^* A^{\mu\,(n)}. \tag{S92}$$

From this we conclude we have:

- One massless real scalar field ϕ.

- One massless real vector field $A_\mu^{(0)}$.

- A tower of massive complex vector fields $A_\mu^{(n)}$ with masses n/R.

By the way, if you do the analogous calculation starting with a five-dimensional metric, the massless degrees of freedom you'll get are a real scalar, a real vector, and a four-dimensional metric, suggesting a unification between electromagnetism and gravity; that's why Einstein was a big fan of this theory.