PHYS-330: QFT I — Problem Set 1 (due 10/4)

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1. Using natural units. (8 points)

In this course, we will work in "natural" units, where $\hbar = k_B = c = \mu_0 = \epsilon_0 = 1$. As a result, any physical quantity $\mathcal{A}$ has the same dimensions as $(1\,\mathrm{eV})^n$ for some n, which we write as $[\mathcal{A}] = n$. For example, we have

$$[\text{energy}] = [\text{momentum}] = [\text{mass}] = [\text{temperature}] = 1 \tag{1}$$

and

$$[\text{length}] = [\text{time}] = -1. \tag{2}$$

These results immediately imply, e.g., [frequency] $= 1$, [speed] $= 0$, and [volume] $= -3$.

If the mass of a particle in natural units is $m = 1\,\mathrm{eV}$, that means its mass in SI units is

$$m = \frac{1\,\mathrm{eV}}{c^2} = 1.8 \times 10^{-36}\,\mathrm{kg}. \tag{3}$$

One physical interpretation is that a particle of this mass has rest energy $1\,\mathrm{eV}$.

a) Find the wavelength and period of a photon of energy $1\,\mathrm{eV}$ in SI units.

b) Express the temperature $T = 1\,\mathrm{eV}$ in SI units.

If you commit the above results to memory, you should always be able to recover numeric values in SI units. The other skill you need is going from SI units to natural units.

c) Find $[G]$, where G is Newton's constant.

d) Find $[n]$, $[P]$, and $[\rho]$, where n is number density, P is pressure, and ρ is mass density.

e) Find $[\phi]$, where ϕ is a real scalar field.

f) Find $[q]$, $[A]$, $[E]$ and $[B]$, where q is electric charge, A is vector potential, and E and B are electric and magnetic fields.

Once you're comfortable with natural units, they'll be an incredibly convenient tool for making rough estimates. For example, the mass of the proton is $m_p \sim \mathrm{GeV}$, and everything in nuclear physics is roughly governed by this scale. From this, we can immediately conclude that the radius of the proton is roughly $r \sim \mathrm{GeV}^{-1}$, in natural units.

g) Write down rough expressions for the density and electric field within a nucleus, and the temperature above which nuclei melt into quark-gluon plasma, in natural units.

Technically, all of the estimates here will be a bit off, because some of these quantities are actually determined by the pion mass $m_\pi \sim 10^{-1}\,\mathrm{GeV}$, as the pion governs the forces between nucleons. For a more careful treatment, see *Astronomical reach of fundamental physics* by Burrows and Ostriker.

2. The harmonic oscillator in quantum mechanics. (15 points)

This exercise reviews the quantum harmonic oscillator, which has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{4}$$

a) Write the Hamiltonian in terms of the ladder operators a and $a^\dagger$, where

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\,x + i\,\frac{p}{\sqrt{m\omega}}\right). \tag{5}$$

b) The normalized vacuum state $|0\rangle$ is defined to satisfy $a|0\rangle = 0$. The number states are then defined by $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ for any integer n. Show that $[a, a^\dagger] = 1$, and use this fact alone to show that the number states are properly normalized.

c) Calculate the expectation values of x, p, and the number operator $N = a^\dagger a$ in the number state $|n\rangle$.

d) Calculate the standard deviations Δx, Δp and ΔN in the number state $|n\rangle$. For what n is the Heisenberg uncertainty product $\Delta x\,\Delta p$ minimal?

e) Suppose the particle begins in the vacuum state $|0\rangle$, and at time $t = 0$, we apply an impulse α. This can be modeled by a Hamiltonian term $-\alpha x \delta(t)$, and the state immediately after the impulse is

$$|\alpha\rangle = e^{i\alpha x}|0\rangle. \tag{6}$$

Show that $|\alpha\rangle$ is an eigenvector of a, and find the eigenvalue.

f) Find the expectation values of x, p, and N, and their standard deviations, for all $t > 0$. (Hint: after you find the answers for the initial state $|\alpha\rangle$, it is easiest to generalize to arbitrary t using Heisenberg picture.)

Your result in part (e) shows that $|\alpha\rangle$ is a so-called coherent state. You might have heard that they are important because they are the "most classical" states. A more important reason is that they are the states you automatically get when you drive a quantum system. As you can see from your results, in the limit of strong driving, Δx and Δp become negligible compared to x and p, and we recover classical physics. Later we will see how a similar result allows quantum fields to behave like classical fields.

3. The relativistic classical point particle. (12 points)

The spacetime trajectory of a relativistic point particle is $x^\mu(\tau) = (x^0(\tau), \mathbf{x}(\tau))$, where τ is an arbitrary parameter. The corresponding action is proportional to the relativistic "length" of the trajectory, where the relativistic line element is

$$ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2. \tag{7}$$

The action is therefore

$$S = -\alpha \int_{\mathcal{P}} ds = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{8}$$

where α is a constant, and τ_1 and τ_2 are the initial and final values of the parameter.

a) The easiest way to understand the nonrelativistic limit $|\partial_t x^i| \ll 1$ is to set $\tau = t$. By demanding that the action reduces to that of a free nonrelativistic particle of mass m (plus a constant), determine the value of the constant α.

If we don't just set $\tau = t$, there are four Euler–Lagrange equations and canonical momenta,

$$\frac{dp^\mu}{d\tau} = \frac{\partial L}{\partial x_\mu}, \quad p^\mu = \frac{\partial L}{\partial(dx_\mu/d\tau)}. \tag{9}$$

b) Find the Euler–Lagrange equations for a general parameter τ, then show that they are equivalent to the conservation of the physical four-momentum of the particle.

c) A simple local, Lorentz invariant way to include a force on the particle is to add

$$S_{\text{int}} = -q \int_{\mathcal{P}} A_\mu(x^\mu)\, dx^\mu = -q \int_{\tau_1}^{\tau_2} A_\mu(x^\mu)\, \frac{dx^\mu}{d\tau}\, d\tau \tag{10}$$

to the action, where $A^\mu(x^\mu)$ is a given four-vector field. Calculate p^μ and $\partial L/\partial x^\mu$, continuing to assume general τ.

d) Now set τ to be the proper time s experienced by the particle (so that $ds = d\tau$) and evaluate the Euler–Lagrange equations, simplifying as much as possible.

A warning: if you set τ to proper time *before* doing part (c), and apply the Euler–Lagrange equations anyway, you'll get nonsense. The reason is that the derivation of the Euler–Lagrange equation assumes all the variables $x^\mu(\tau)$ can be varied independently, but when $d\tau = ds$ we automatically have the constraint $\sqrt{\eta_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau)} = 1$. Lagrangians with constraints are subtle and important, but beyond the scope of this course. For much more about them, see *Quantization of Gauge Systems* by Henneaux and Teitelboim.

4. The complex scalar field. (5 points)

The Lagrangian density for a canonically normalized free real scalar field of mass m is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)\,(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{11}$$

Now consider a theory of two free real scalar fields ϕ_1 and ϕ_2, both with mass m.

a) Write their Lagrangian density in terms of the complex scalar field $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its complex conjugate Φ^*.

Complex fields are always equivalent to a pair of equal mass real fields, and are useful because such pairs occur frequently in nature, for reasons we'll see later. (At low energies, we actually don't know of any complex scalar fields, but the electron is described by a Dirac field, which is a complex fermion field built from two equal mass real fermion fields.)

Complex fields are convenient once you get to know them, but they come with an annoying problem: it is not obvious how to vary the action with respect to Φ, because any change in Φ also changes Φ^*. It turns out that you will always get the right results (i.e. results that are equivalent to what you'd get working in terms of the two real fields) by treating Φ and Φ^* as if they were *independent* real fields, even though they clearly aren't. (For an explanation why, see page 56 of Sidney Coleman's lecture notes.)

b) Compute the conjugate momenta Π and Π^* of Φ and Φ^*, and the Euler–Lagrange equations for Φ and Φ^*.

c) Show that the action is invariant under the transformation $\Phi \to e^{i\alpha}\Phi$, for any real α. What is the equivalent symmetry in terms of the real scalar fields ϕ_1 and ϕ_2?