## 1. Using natural units. (8 points)

In this course, we will work in "natural" units, where $\hbar=k_{B}=c=\mu_{0}=\epsilon_{0}=1$. As a result, any physical quantity $\mathcal{A}$ has the same dimensions as $(1 \mathrm{eV})^{n}$ for some $n$, which we write as $[\mathcal{A}]=n$. For example, we have

$$
\begin{equation*}
[\text { energy }]=[\text { momentum }]=[\text { mass }]=[\text { temperature }]=1 \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
[\text { length }]=[\text { time }]=-1 \tag{2}
\end{equation*}
$$

These results immediately imply, e.g., $[$ frequency $]=1$, $[$ speed $]=0$, and $[$ volume $]=-3$.
If the mass of a particle in natural units is $m=1 \mathrm{eV}$, that means its mass in SI units is

$$
\begin{equation*}
m=\frac{1 \mathrm{eV}}{c^{2}}=1.8 \times 10^{-36} \mathrm{~kg} \tag{3}
\end{equation*}
$$

One physical interpretation is that a particle of this mass has rest energy 1 eV .
a) Find the wavelength and period of a photon of energy 1 eV in SI units.

Solution: Since $E=\hbar \omega$, the period is

$$
\begin{equation*}
T=\frac{2 \pi \hbar}{1 \mathrm{eV}}=4 \times 10^{-15} \mathrm{~s} \tag{S1}
\end{equation*}
$$

Since $\lambda=c T$, the wavelength is

$$
\begin{equation*}
\lambda=\frac{2 \pi \hbar c}{1 \mathrm{eV}}=1.2 \times 10^{-6} \mathrm{~m} . \tag{S2}
\end{equation*}
$$

b) Express the temperature $T=1 \mathrm{eV}$ in SI units.

Solution: Since energy has dimensions of $k_{B} T$, the temperature is

$$
\begin{equation*}
T=\frac{1 \mathrm{eV}}{k_{B}}=12000 \mathrm{~K} . \tag{S3}
\end{equation*}
$$

This is the temperature where the typical thermal energy of a degree of freedom is $\sim 1 \mathrm{eV}$, which would be sufficient to break apart molecules and ionize most atoms.

If you commit the above results to memory, you should always be able to recover numeric values in SI units. The other skill you need is going from SI units to natural units.
c) Find $[G]$, where $G$ is Newton's constant.

Solution: You can do this part, and all the parts below, by brute force. But it's faster and more fun to get the answers by thinking about equations you already know.

For example, the gravitational potential energy formula implies $[E]=\left[G M^{2} / R\right]$, so $[G]=[R / M]=-2$. (In natural units, $G \sim 1 / M_{\mathrm{pl}}^{2}$ where $M_{\mathrm{pl}}$ is the Planck mass.)
d) Find $[n],[P]$, and $[\rho]$, where $n$ is number density, $P$ is pressure, and $\rho$ is mass density.

Solution: Since $n$ is a dimensionless number per volume, $[n]=[1 /$ volume $]=3$. Noting that $d U=P d V$ for thermodynamic work, $[P]=[$ energy $/$ volume $]=4$. Similarly, $[\rho]=[$ mass $/$ volume $]=4$.
e) Find $[\phi]$, where $\phi$ is a real scalar field.

Solution: Note that the action $S$ is the integral of a Lagrangian (with units of energy) over time, so

$$
\begin{equation*}
[S]=[\text { energy } \cdot \text { time }]=0 . \tag{S4}
\end{equation*}
$$

In field theory, the action is the integral over 4-dimensional spacetime of a Lagrangian density $\mathcal{L}$, which implies $[\mathcal{L}]=4$. Examining the kinetic term, we have

$$
\begin{equation*}
\left[\left(\partial_{\mu} \phi\right)^{2}\right]=4 \tag{S5}
\end{equation*}
$$

and we have $\left[\partial_{\mu}\right]=1$, which implies $[\phi]=1$.
f) Find $[q],[A],[E]$ and $[B]$, where $q$ is electric charge, $A$ is vector potential, and $E$ and $B$ are electric and magnetic fields.
Solution: In natural units, the force between charges is $F=q^{2} / 4 \pi r^{2}$. Since $[F]=[d p / d t]=2$, we conclude $[q]=0$, i.e. charge is dimensionless. (In case you were wondering, the specific value of the electron charge is about -0.3 . This result is more commonly expressed in terms of the "fine structure constant" $\alpha=e^{2} / 4 \pi \approx 1 / 137$, and later, the fact that $\alpha \ll 1$ will tell us that electromagnetic interactions in QFT can be treated in perturbation theory.)

For the vector potential, one could recall the Lorenz gauge equation of motion $\partial^{2} A^{\mu}=J^{\mu}$ where $J^{\mu}$ is the current density. We have $\left[\partial^{2}\right]=2$ and $[J]=[$ charge $/$ volume $]=3$, so we conclude $[A]=1$.

The electric and magnetic fields are spatial or time derivatives of the vector potential, so $[E]=[B]=2$.
Once you're comfortable with natural units, they'll be an incredibly convenient tool for making rough estimates. For example, the mass of the proton is $m_{p} \sim \mathrm{GeV}$, and everything in nuclear physics is roughly governed by this scale. From this, we can immediately conclude that the radius of the proton is roughly $r \sim \mathrm{GeV}^{-1}$, in natural units.
g) Write down rough expressions for the density and electric field within a nucleus, and the temperature above which nuclei melt into quark-gluon plasma, in natural units.

Solution: It's just $\rho \sim \mathrm{GeV}^{4}, E \sim \mathrm{GeV}^{2}$, and $T \sim \mathrm{GeV}$. These are all very rough estimates, dropping dimensionless factors like $\alpha \sim 1 / 137$ or dependence on the pion mass, but they're all within a few orders of magnitude.

Technically, all of the estimates here will be a bit off, because some of these quantities are actually determined by the pion mass $m_{\pi} \sim 10^{-1} \mathrm{GeV}$, as the pion governs the forces between nucleons. For a more careful treatment, see Astronomical reach of fundamental physics by Burrows and Ostriker.

## 2. The harmonic oscillator in quantum mechanics. (15 points)

This exercise reviews the quantum harmonic oscillator, which has Hamiltonian

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}+\frac{m \omega^{2} x^{2}}{2} \tag{4}
\end{equation*}
$$

a) Write the Hamiltonian in terms of the ladder operators $a$ and $a^{\dagger}$, where

$$
\begin{equation*}
a=\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} x+i \frac{p}{\sqrt{m \omega}}\right) \tag{5}
\end{equation*}
$$

Solution: Of course, we have

$$
\begin{equation*}
a^{\dagger}=\frac{1}{\sqrt{2}}\left(\sqrt{m \omega} x-i \frac{p}{\sqrt{m \omega}}\right) \tag{S6}
\end{equation*}
$$

Following the usual textbook steps, you should find

$$
\begin{equation*}
H=\omega\left(a^{\dagger} a+\frac{1}{2}\right) . \tag{S7}
\end{equation*}
$$

Equivalent forms, such as having $a^{\dagger} a+a a^{\dagger}$ in parentheses, are also acceptable.
b) The normalized vacuum state $|0\rangle$ is defined to satisfy $a|0\rangle=0$. The number states are then defined by $a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle$ for any integer $n$. Show that $\left[a, a^{\dagger}\right]=1$, and use this fact alone to show that the number states are properly normalized.

Solution: Using the fact that $[x, p]=i$, we have

$$
\begin{align*}
{\left[a, a^{\dagger}\right] } & =\frac{1}{2}\left[\sqrt{m \omega} x+i \frac{p}{\sqrt{m \omega}}, \sqrt{m \omega} x-i \frac{p}{\sqrt{m \omega}}\right]  \tag{S8}\\
& =\frac{1}{2}([x,-i p]+[i p, x])=1 \tag{S9}
\end{align*}
$$

as desired.

Now, by repeatedly using the definition of the number states, we have

$$
\begin{equation*}
|n\rangle=\frac{1}{\sqrt{n!}}\left(a^{\dagger}\right)^{n}|0\rangle \tag{S10}
\end{equation*}
$$

Note that we have

$$
\begin{equation*}
\left[a,\left(a^{\dagger}\right)^{n}\right]=n\left(a^{\dagger}\right)^{n-1} . \tag{S11}
\end{equation*}
$$

The intuition is that to convert $a\left(a^{\dagger}\right)^{n}$ into $\left(a^{\dagger}\right)^{n} a$, we need to move an $a$ through an $a^{\dagger}$ a total of $n$ times. For each move, we use the identity $a a^{\dagger}=a^{\dagger} a+\left[a, a^{\dagger}\right]=a^{\dagger} a+1$, and therefore pick up a factor of $\left(a^{\dagger}\right)^{n-1}$. The factor of $n$ in the final result is because we need to do $n$ such moves.

To check the normalization, we just compute

$$
\begin{align*}
\langle n \mid n\rangle & =\frac{1}{n!}\langle 0| a^{n}\left(a^{\dagger}\right)^{n}|0\rangle  \tag{S12}\\
& =\frac{1}{n!}\langle 0| a^{n-1} a\left(a^{\dagger}\right)^{n}|0\rangle  \tag{S13}\\
& =\frac{1}{n!}\langle 0| a^{n-1}\left(\left(a^{\dagger}\right)^{n} a+n\left(a^{\dagger}\right)^{n-1}\right)|0\rangle  \tag{S14}\\
& =\frac{1}{(n-1)!}\langle 0| a^{n-1}\left(a^{\dagger}\right)^{n-1}|0\rangle  \tag{S15}\\
& =\langle n-1 \mid n-1\rangle . \tag{S16}
\end{align*}
$$

This shows that the norms of all the number states are equal, so they're all normalized because $|0\rangle$ is.
c) Calculate the expectation values of $x, p$, and the number operator $N=a^{\dagger} a$ in the number state $|n\rangle$.

Solution: We can write $x$ and $p$ in terms of ladder operators as

$$
\begin{equation*}
x=\frac{1}{\sqrt{2 m \omega}}\left(a^{\dagger}+a\right), \quad p=i \sqrt{\frac{m \omega}{2}}\left(a^{\dagger}-a\right) . \tag{S17}
\end{equation*}
$$

Then $\langle n| x|n\rangle$ has two terms, one proportional to $\langle n \mid n+1\rangle$ and the other proportional to $\langle n \mid n-1\rangle$, which both vanish. Thus, $\langle n| x|n\rangle=0$, and by similar reasoning $\langle n| p|n\rangle=0$. As for the number operator,

$$
\begin{equation*}
\langle n| N|n\rangle=\langle n| a^{\dagger} a|n\rangle=\langle n-1| \sqrt{n} \sqrt{n}|n-1\rangle=n . \tag{S18}
\end{equation*}
$$

d) Calculate the standard deviations $\Delta x, \Delta p$ and $\Delta N$ in the number state $|n\rangle$. For what $n$ is the Heisenberg uncertainty product $\Delta x \Delta p$ minimal?

Solution: For a given state, the standard deviation of any operator $A$ is defined by

$$
\begin{equation*}
\Delta A=\sqrt{\left\langle A^{2}\right\rangle-\langle A\rangle^{2}} \tag{S19}
\end{equation*}
$$

where the expectation values are taken with respect to the state. Now first, we note that

$$
\begin{align*}
\langle n| x^{2}|n\rangle & =\frac{1}{2 m \omega}\langle n|\left(a+a^{\dagger}\right)\left(a+a^{\dagger}\right)|n\rangle  \tag{S20}\\
& =\frac{1}{2 m \omega}\langle n| a^{\dagger} a+a a^{\dagger}|n\rangle  \tag{S21}\\
& =\frac{1}{2 m \omega}(n+(n+1))  \tag{S22}\\
& =\frac{2 n+1}{2 m \omega} . \tag{S23}
\end{align*}
$$

From this and the previous part, we conclude

$$
\begin{equation*}
\Delta x=\sqrt{\frac{2 n+1}{2 m \omega}} \tag{S24}
\end{equation*}
$$

A very similar computation for $p$ gives

$$
\begin{equation*}
\Delta p=\sqrt{\frac{m \omega(2 n+1)}{2}} \tag{S25}
\end{equation*}
$$

Clearly the Heisenberg uncertainty product is minimal for $n=0$, in which case $\Delta x \Delta p=1 / 2$, saturating the uncertainty relation.

Finally, we have

$$
\begin{equation*}
\langle n| N^{2}|n\rangle=n\langle n| N|n\rangle=n^{2} \tag{S26}
\end{equation*}
$$

from which we conclude $\Delta N=0$, as expected.
e) Suppose the particle begins in the vacuum state $|0\rangle$, and at time $t=0$, we apply an impulse $\alpha$. This can be modeled by a Hamiltonian term $-\alpha x \delta(t)$, and the state immediately after the impulse is

$$
\begin{equation*}
|\alpha\rangle=e^{i \alpha x}|0\rangle \tag{6}
\end{equation*}
$$

Show that $|\alpha\rangle$ is an eigenvector of $a$, and find the eigenvalue.
Solution: For simplicity, let's define $\beta=\alpha / \sqrt{2 m \omega}$, so that

$$
\begin{equation*}
|\alpha\rangle=e^{i \beta\left(a+a^{\dagger}\right)}|0\rangle . \tag{S27}
\end{equation*}
$$

We'll show a direct solution. First, note that $\left[a, a+a^{\dagger}\right]=1$. This implies that

$$
\begin{equation*}
\left[a,\left(a+a^{\dagger}\right)^{n}\right]=n\left(a+a^{\dagger}\right)^{n-1} \tag{S28}
\end{equation*}
$$

where the factor of $n$ comes from commuting the $a$ past each copy of $a+a^{\dagger}$. Now we have

$$
\begin{align*}
a|\alpha\rangle & =a \sum_{n=0}^{\infty} \frac{(i \beta)^{n}}{n!}\left(a+a^{\dagger}\right)^{n}|0\rangle  \tag{S29}\\
& =\sum_{n=0}^{\infty} \frac{(i \beta)^{n}}{n!}\left(\left[a,\left(a+a^{\dagger}\right)^{n}\right]+\left(a+a^{\dagger}\right)^{n} a\right)|0\rangle  \tag{S30}\\
& =\sum_{n=0}^{\infty} \frac{(i \beta)^{n}}{n!}\left(n\left(a+a^{\dagger}\right)^{n-1}\right)|0\rangle  \tag{S31}\\
& =\sum_{n=0}^{\infty} \frac{(i \beta)^{n}}{(n-1)!}\left(a+a^{\dagger}\right)^{n-1}|0\rangle  \tag{S32}\\
& =\sum_{n=0}^{\infty} \frac{(i \beta)^{n+1}}{n!}\left(a+a^{\dagger}\right)^{n}|0\rangle  \tag{S33}\\
& =i \beta|\alpha\rangle \tag{S34}
\end{align*}
$$

where we Taylor expanded the exponential, used the definition of the commutator, used $a|0\rangle=0$ and Eq. (S28), and shifted the variable $n$. Thus, $|\alpha\rangle$ is an eigenvector of $a$ with eigenvalue $i \beta=i \alpha / \sqrt{2 m \omega}$.
f) Find the expectation values of $x, p$, and $N$, and their standard deviations, for all $t>0$. (Hint: after you find the answers for the initial state $|\alpha\rangle$, it is easiest to generalize to arbitrary $t$ using Heisenberg picture.)

Solution: First, let's calculate the expectation values in the initial state $|\alpha\rangle$. Since $x$ commutes with $e^{i \alpha x}$, we have

$$
\begin{equation*}
\langle\alpha| x|\alpha\rangle=\langle 0| e^{-i \alpha x} x e^{i \alpha x}|0\rangle=\langle 0| x|0\rangle=0 \tag{S35}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\langle\alpha| x^{2}|\alpha\rangle=\langle 0| x^{2}|0\rangle=\frac{1}{2 m \omega} \tag{S36}
\end{equation*}
$$

where we used Eq. S24. This gives a standard deviation

$$
\begin{equation*}
\Delta x=\frac{1}{\sqrt{2 m \omega}} \tag{S37}
\end{equation*}
$$

Next, for $p$ it's useful to use the decomposition into ladder operators, as

$$
\begin{equation*}
\langle\alpha| a|\alpha\rangle=\langle\alpha| i \beta|\alpha\rangle=i \beta, \quad\langle\alpha| a^{\dagger}|\alpha\rangle=(a|\alpha\rangle)^{\dagger}|\alpha\rangle=-i \beta \tag{S38}
\end{equation*}
$$

where we used the result of part (e). We therefore have

$$
\begin{equation*}
\langle\alpha| p|\alpha\rangle=i \sqrt{\frac{m \omega}{2}}\langle\alpha|\left(a^{\dagger}-a\right)|\alpha\rangle=(-2 i \beta) i \sqrt{\frac{m \omega}{2}}=\alpha \tag{S39}
\end{equation*}
$$

which is unsurprising, as the state $|\alpha\rangle$ as defined by exerting an impulse $\alpha$ on the ground state. Next,

$$
\begin{align*}
\langle\alpha| p^{2}|\alpha\rangle & =-\frac{m \omega}{2}\langle\alpha|\left(a^{\dagger}-a\right)^{2}|\alpha\rangle  \tag{S40}\\
& =-\frac{m \omega}{2}\langle\alpha|\left(a^{\dagger} a^{\dagger}+a a-a a^{\dagger}-a^{\dagger} a\right)|\alpha\rangle  \tag{S41}\\
& =\frac{m \omega}{2}\langle\alpha|\left(-a^{\dagger} a^{\dagger}-a a+2 a^{\dagger} a+1\right)|\alpha\rangle  \tag{S42}\\
& =\frac{m \omega}{2}\left(4 \beta^{2}+1\right)  \tag{S43}\\
& =\frac{m \omega}{2}+\alpha^{2} . \tag{S44}
\end{align*}
$$

Therefore, the standard deviation is

$$
\begin{equation*}
\Delta p=\sqrt{\frac{m \omega}{2}} \tag{S45}
\end{equation*}
$$

which is unchanged from the ground state. Finally, for the number states,

$$
\begin{equation*}
\langle\alpha| N|\alpha\rangle=\langle\alpha| a^{\dagger} a|\alpha\rangle=\beta^{2}=\frac{\alpha^{2}}{2 m \omega} \tag{S46}
\end{equation*}
$$

and

$$
\begin{align*}
\langle\alpha| N^{2}|\alpha\rangle & =\langle\alpha| a^{\dagger} a a^{\dagger} a|\alpha\rangle  \tag{S47}\\
& =\beta^{2}\langle\alpha| a a^{\dagger}|\alpha\rangle  \tag{S48}\\
& =\beta^{2}\langle\alpha|\left(1+a^{\dagger} a\right)|\alpha\rangle  \tag{S49}\\
& =\beta^{2}+\beta^{4} \tag{S50}
\end{align*}
$$

from which we conclude

$$
\begin{equation*}
\Delta N=\beta=\frac{\alpha}{\sqrt{2 m \omega}} \tag{S51}
\end{equation*}
$$

You could also derive all of the above results using the Baker-Campbell-Hausdorff theorem, but such machinery isn't necessary. We simply leaned on the result of part (e) to get all of the results, and the
solution to part (e) is effectively a proof of that theorem in the simple case we need.
Now let's consider how these expectation values and uncertainties evolve over time. The evolution of an expectation value of any operator $\mathcal{O}$ is

$$
\begin{equation*}
\langle\mathcal{O}\rangle(t)=\langle\psi| e^{i H t} \mathcal{O} e^{-i H t}|\psi\rangle . \tag{S52}
\end{equation*}
$$

Since $N$ commutes with $H$, this implies that $N$ and $\Delta N$ just stay constant over time. As for $x$ and $p$, the easiest way is to compute the "Heisenberg" operators $\mathcal{O}_{H}(t)=e^{i H t} \mathcal{O} e^{-i H t}$. The basic result is that Heisenberg operators satisfy Hamilton's equations, which here imply

$$
\begin{equation*}
\frac{d p_{H}}{d t}=-m \omega^{2} x_{H}, \quad \frac{d x_{H}}{d t}=\frac{p_{H}}{m} . \tag{S53}
\end{equation*}
$$

The solution of this pair of equations is

$$
\begin{equation*}
p_{H}(t)=p \cos \omega t-m \omega x \sin \omega t, \quad x_{H}(t)=x \cos \omega t+\frac{p}{m \omega} \sin \omega t . \tag{S54}
\end{equation*}
$$

This means the expectation values we're after follow immediately from those we already calculated. We know that $\langle\alpha| x|\alpha\rangle=0$ and $\langle\alpha| p|\alpha\rangle=\alpha$, from which we conclude

$$
\begin{equation*}
\langle x\rangle(t)=\frac{\alpha}{m \omega} \sin \omega t, \quad\langle p\rangle(t)=\alpha \cos \omega t \tag{S55}
\end{equation*}
$$

Similarly, we have

$$
\begin{align*}
\left\langle x^{2}\right\rangle(t) & =\langle\alpha| x^{2} \cos ^{2} \omega t+\frac{(x p+p x)}{m \omega} \cos \omega t \sin \omega t+\frac{p^{2}}{m^{2} \omega^{2}} \sin ^{2} \omega t|\alpha\rangle  \tag{S56}\\
& =\cos ^{2} \omega t\langle\alpha| x^{2}|\alpha\rangle+\frac{\sin ^{2} \omega t}{m^{2} \omega^{2}}\langle\alpha| p^{2}|\alpha\rangle  \tag{S57}\\
& =\left(\frac{\alpha \sin \omega t}{m \omega}\right)^{2}+\frac{1}{2 m \omega} \tag{S58}
\end{align*}
$$

which implies that the position uncertainty does not change over time,

$$
\begin{equation*}
\Delta x(t)=\frac{1}{\sqrt{2 m \omega}} \tag{S59}
\end{equation*}
$$

By an extremely similar computation, we find the momentum uncertainty doesn't change over time either,

$$
\begin{equation*}
\Delta p(t)=\sqrt{\frac{m \omega}{2}} . \tag{S60}
\end{equation*}
$$

Your result in part (e) shows that $|\alpha\rangle$ is a so-called coherent state. You might have heard that they are important because they are the "most classical" states. A more important reason is that they are the states you automatically get when you drive a quantum system. As you can see from your results, in the limit of strong driving, $\Delta x$ and $\Delta p$ become negligible compared to $x$ and $p$, and we recover classical physics. Later we will see how a similar result allows quantum fields to behave like classical fields.

## 3. The relativistic classical point particle. (12 points)

The spacetime trajectory of a relativistic point particle is $x^{\mu}(\tau)=\left(x^{0}(\tau), \mathbf{x}(\tau)\right)$, where $\tau$ is an arbitrary parameter. The corresponding action is proportional to the relativistic "length" of the trajectory, where the relativistic line element is

$$
\begin{equation*}
d s^{2}=\eta_{\mu \nu} d x^{\mu} d x^{\nu}=d t^{2}-d x^{2}-d y^{2}-d z^{2} . \tag{7}
\end{equation*}
$$

The action is therefore

$$
\begin{equation*}
S=-\alpha \int_{\mathcal{P}} d s=-\alpha \int_{\tau_{1}}^{\tau_{2}} d \tau \sqrt{\eta_{\mu \nu} \frac{d x^{\mu}}{d \tau} \frac{d x^{\nu}}{d \tau}} \tag{8}
\end{equation*}
$$

where $\alpha$ is a constant, and $\tau_{1}$ and $\tau_{2}$ are the initial and final values of the parameter.
a) The easiest way to understand the nonrelativistic limit $\left|\partial_{t} x^{i}\right| \ll 1$ is to set $\tau=t$. By demanding that the action reduces to that of a free nonrelativistic particle of mass $m$ (plus a constant), determine the value of the constant $\alpha$.

Solution: Setting $\tau=t$, we have

$$
\begin{equation*}
S=-\alpha \int d t \sqrt{\eta_{\mu \nu} \frac{d x^{\mu}}{d t} \frac{d x^{\nu}}{d t}}=-\alpha \int d t \sqrt{1-v(t)^{2}} \tag{S61}
\end{equation*}
$$

where $v(t)$ is the ordinary speed. In the nonrelativistic limit $v \ll 1$,

$$
\begin{equation*}
S \approx-\alpha \int d t\left(1-\frac{v^{2}}{2}+O\left(v^{4}\right)\right) \tag{S62}
\end{equation*}
$$

The first term is just an irrelevant constant, while the next gives a Lagrangian $\alpha v^{2} / 2$. On the other hand, we know the Lagrangian for a free nonrelativistic particle is $m v^{2} / 2$, so we conclude $\alpha=m$.

If we don't just set $\tau=t$, there are four Euler-Lagrange equations and canonical momenta,

$$
\begin{equation*}
\frac{d p^{\mu}}{d \tau}=\frac{\partial L}{\partial x_{\mu}}, \quad p^{\mu}=\frac{\partial L}{\partial\left(d x_{\mu} / d \tau\right)} \tag{9}
\end{equation*}
$$

b) Find the Euler-Lagrange equations for a general parameter $\tau$, then show that they are equivalent to the conservation of the physical four-momentum of the particle.
Solution: For simplicity, let a dot denote a derivative with respect to $\tau$, so that

$$
\begin{equation*}
L=-m \sqrt{\eta_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\nu}}=-m \sqrt{\dot{x}^{2}} \tag{S63}
\end{equation*}
$$

The canonical momenta are

$$
\begin{equation*}
p^{\mu}=\frac{\partial L}{\partial \dot{x}_{\mu}}=-\frac{m}{2 \sqrt{\dot{x}^{2}}}\left(2 \dot{x}^{\mu}\right)=-\frac{m \dot{x}^{\mu}}{\sqrt{\dot{x}^{2}}} \tag{S64}
\end{equation*}
$$

The Lagrangian has no direct dependence on $x^{\mu}$, so the Euler-Lagrange equations just state that $p^{\mu}$ is conserved, $d p^{\mu} / d \tau=0$.

This isn't the end of the problem, though; we still need to show how the canonical momentum $p^{\mu}$ is related to physical four-momentum. One way to do this is to let $\tau$ be the proper time $s$ of the particle. Then $\dot{x}^{2}=1$, and $p^{\mu}=-m d x^{\mu} / d s$-momentum. Alternatively, we can let $\tau$ be the coordinate time $t$, in which case $p^{\mu}=-m\left(d x^{\mu} / d t\right) / \sqrt{1-v^{2}}$. In both cases, it's clear that $p^{\mu}$ is the negative of the physical four-momentum. (This negative sign is weird, but harmless, since the physical results are the same. It occurs because we used a mostly negative metric, which will pay off later in the course.)
c) A simple local, Lorentz invariant way to include a force on the particle is to add

$$
\begin{equation*}
S_{\mathrm{int}}=-q \int_{\mathcal{P}} A_{\mu}\left(x^{\mu}\right) d x^{\mu}=-q \int_{\tau_{1}}^{\tau_{2}} A_{\mu}\left(x^{\mu}\right) \frac{d x^{\mu}}{d \tau} d \tau \tag{10}
\end{equation*}
$$

to the action, where $A^{\mu}\left(x^{\mu}\right)$ is a given four-vector field. Calculate $p^{\mu}$ and $\partial L / \partial x^{\mu}$, continuing to assume general $\tau$.
Solution: There are a lot of $\mu$ 's floating around, so to avoid confusion, let's use the fact that we can always rename a dummy index to rewrite the interaction as

$$
\begin{equation*}
S_{\mathrm{int}}=-q \int A_{\nu}\left(x^{\mu}\right) \dot{x}^{\nu} d \tau \tag{S65}
\end{equation*}
$$

First, the canonical momenta are changed to

$$
\begin{equation*}
p^{\mu}=\frac{\partial L}{\partial \dot{x}_{\mu}}=-\frac{m \dot{x}^{\mu}}{\sqrt{\dot{x}^{2}}}-q A^{\mu}\left(x^{\mu}\right) . \tag{S66}
\end{equation*}
$$

Next, the right-hand side of the Euler-Lagrange equations is now nontrivial,

$$
\begin{equation*}
\frac{\partial L}{\partial x_{\mu}}=-q \dot{x}^{\nu} \partial^{\mu} A_{\nu}\left(x^{\mu}\right) \tag{S67}
\end{equation*}
$$

where the partial derivative on the right is with respect to $x^{\mu}$.
A common issue with this part was mismatched indices (having $\mu$ up on one side and down on the other side), which can be avoided by taking care to make the index positions consistent at each step. Another issue was duplicated indices (having more than two copies of $\mu$, which is meaningless), which can be avoided by renaming dummy indices when required.
d) Now set $\tau$ to be the proper time $s$ experienced by the particle (so that $d s=d \tau$ ) and evaluate the Euler-Lagrange equations, simplifying as much as possible.

Solution: From this point on, we'll leave the $x^{\mu}$ argument of the vector potential implicit. Specializing the proper time, the canonical momenta are

$$
\begin{equation*}
p^{\mu}=-m \dot{x}^{\mu}-q A^{\mu} . \tag{S68}
\end{equation*}
$$

We therefore have

$$
\begin{equation*}
\frac{d p^{\mu}}{d s}=-m \ddot{x}^{\mu}-q \frac{d A^{\mu}}{d s}=-m \ddot{x}^{\mu}-q \dot{x}^{\nu} \partial_{\nu} A^{\mu} \tag{S69}
\end{equation*}
$$

Equating this with Eq. (S67) gives

$$
\begin{equation*}
m \ddot{x}^{\mu}=-q \dot{x}^{\nu}\left(\partial_{\nu} A^{\mu}-\partial^{\mu} A_{\nu}\right)=q \dot{x}_{\nu}\left(\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right) . \tag{S70}
\end{equation*}
$$

This is an acceptable final answer; the right-hand side can also be written as $q \dot{x}_{\nu} F^{\mu \nu}$ where $F$ is the electromagnetic field strength tensor. If you treat $A^{\mu}$ as a vector potential and plug in the definitions of the electric and magnetic fields, you'll recover the Lorentz force law.

A warning: if you set $\tau$ to proper time before doing part (c), and apply the Euler-Lagrange equations anyway, you'll get nonsense. The reason is that the derivation of the EulerLagrange equation assumes all the variables $x^{\mu}(\tau)$ can be varied independently, but when $d \tau=d s$ we automatically have the constraint $\sqrt{\eta_{\mu \nu}\left(d x^{\mu} / d \tau\right)\left(d x^{\nu} / d \tau\right)}=1$. Lagrangians with constraints are subtle and important, but beyond the scope of this course. For much more about them, see Quantization of Gauge Systems by Henneaux and Teitelboim.

## 4. The complex scalar field. (5 points)

The Lagrangian density for a canonically normalized free real scalar field of mass $m$ is

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2}\left(\partial_{\mu} \phi\right)\left(\partial^{\mu} \phi\right)-\frac{1}{2} m^{2} \phi^{2} . \tag{11}
\end{equation*}
$$

Now consider a theory of two free real scalar fields $\phi_{1}$ and $\phi_{2}$, both with mass $m$.
a) Write their Lagrangian density in terms of the complex scalar field $\Phi=\left(\phi_{1}+i \phi_{2}\right) / \sqrt{2}$ and its complex conjugate $\Phi^{*}$.

Solution: The mass terms involve $\phi_{1}^{2}+\phi_{2}^{2}$, which clearly indicates the answer involves the complex norm $\Phi^{*} \Phi$. Treating the kinetic term the same way, and fixing the numeric constants, gives

$$
\begin{equation*}
\mathcal{L}=\left(\partial_{\mu} \Phi^{*}\right)\left(\partial^{\mu} \Phi\right)-m^{2} \Phi^{*} \Phi . \tag{S71}
\end{equation*}
$$

Complex fields are always equivalent to a pair of equal mass real fields, and are useful because such pairs occur frequently in nature, for reasons we'll see later. (At low energies, we actually don't know of any complex scalar fields, but the electron is described by a Dirac field, which is a complex fermion field built from two equal mass real fermion fields.)

Complex fields are convenient once you get to know them, but they come with an annoying problem: it is not obvious how to vary the action with respect to $\Phi$, because any change in $\Phi$ also changes $\Phi^{*}$. It turns out that you will always get the right results (i.e. results that are equivalent to what you'd get working in terms of the two real fields) by treating $\Phi$ and $\Phi^{*}$ as if they were independent real fields, even though they clearly aren't. (For an explanation why, see page 56 of Sidney Coleman's lecture notes.)
b) Compute the conjugate momenta $\Pi$ and $\Pi^{*}$ of $\Phi$ and $\Phi^{*}$, and the Euler-Lagrange equations for $\Phi$ and $\Phi^{*}$.
Solution: Note that the kinetic term contains $\dot{\Phi}^{*} \dot{\Phi}$, where a dot denotes a time derivative. Thus, the canonical momenta are

$$
\begin{equation*}
\Pi=\frac{\partial \mathcal{L}}{\partial \dot{\Phi}}=\dot{\Phi}^{*}, \quad \Pi^{*}=\frac{\partial \mathcal{L}}{\partial \dot{\Phi}^{*}}=\dot{\Phi} \tag{S72}
\end{equation*}
$$

The Euler-Lagrange equations are

$$
\begin{equation*}
\partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial^{\mu} \Phi\right)}-\frac{\partial \mathcal{L}}{\partial \Phi}=0, \quad \partial_{\mu} \frac{\partial \mathcal{L}}{\partial\left(\partial^{\mu} \Phi^{*}\right)}-\frac{\partial \mathcal{L}}{\partial \Phi^{*}}=0 \tag{S73}
\end{equation*}
$$

from which we read off

$$
\begin{equation*}
\left(\partial^{2}+m^{2}\right) \Phi^{*}=0, \quad\left(\partial^{2}+m^{2}\right) \Phi=0 \tag{S74}
\end{equation*}
$$

which is just the Klein-Gordan equation.
c) Show that the action is invariant under the transformation $\Phi \rightarrow e^{i \alpha} \Phi$, for any real $\alpha$. What is the equivalent symmetry in terms of the real scalar fields $\phi_{1}$ and $\phi_{2}$ ?

Solution: Each term in the Lagrangian has one power of $\Phi$ and one power of $\Phi^{*}$, and $\Phi^{*} \rightarrow e^{-i \alpha} \Phi$, so the phase cancels out. In terms of the real scalar fields, an equivalent symmetry is a rotation,

$$
\begin{equation*}
\phi_{1} \rightarrow \phi_{1} \cos \alpha-\phi_{2} \sin \alpha, \quad \phi_{2} \rightarrow \phi_{2} \cos \alpha+\phi_{1} \sin \alpha \tag{S75}
\end{equation*}
$$

