PHYS-330: QFT I — Problem Set 2 (due 10/11)

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1. Lorentz transformations. (10 points)

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ are linear transformations that leave inner products invariant, meaning that $x^\mu y_\mu = x'^\mu y'_\mu$ for any four-vectors x and y.

a) Show that this implies

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu. \tag{1}$$

b) All proper, orthochronous Lorentz transformations (i.e. all those which preserve the orientation of space and the direction of time) can be decomposed into infinitesimal Lorentz transformations. These take the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon\, \omega^\mu{}_\nu x^\nu \tag{2}$$

where ϵ is infinitesimal. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

c) The elements of any infinitesimal Lorentz transformation $\omega^\mu{}_\nu$ can be written as a 4×4 matrix, where μ and ν index the row and column, respectively. For an infinitesimal rotation $\epsilon = d\theta$ about the z-axis, write out this matrix, and denote it by iJ^3 for later. What exponential of J^3 corresponds to a finite rotation by an angle θ?

d) Write down the matrix iK^1 corresponding to an infinitesimal boost by $\epsilon = dv$ about the x-axis. What exponential of K^1 corresponds to a finite boost by a velocity v?

e) Defining J^1, J^2, K^2, and K^3 similarly, the generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{3}$$

This is the Lie algebra of the Lorentz group. Physically, these results tells us that infinitesimal rotations and boosts are vectors (i.e. angular velocity and velocity are vectors), and that composing boosts can yield a rotation. Prove these results for the cases $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$. (The proofs for other cases are similar.)

2. Quantization of the complex scalar field. (30 points)

This will be an involved problem, but it will teach you everything there is to know about free field mode expansions. (You don't have to write every detail; when it's clear other computations will go the same way as one you just did, it's fine to just say so and move on.) In problem set 1 we considered a complex scalar field Φ with Lagrangian density

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi, \tag{4}$$

and found the canonical fields Φ and Φ^* and momenta Π and Π^*. Letting $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$, introduce the following mode expansion for the canonical fields and momenta:

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{5}$$

$$\Phi^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{6}$$

$$\Pi(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{7}$$

$$\Pi^*(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{8}$$

We quantize the fields by imposing the canonical commutation relations

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = [\Phi^*(\mathbf{x}), \Pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{9}$$

with all other commutators between these fields vanishing. (Technically, we should write Hermitian conjugates $\Phi^\dagger$ and $\Pi^\dagger$ here, since Φ and Π are now operators, but we'll continue to use stars to emphasize the links between the classical and quantum theories.)

a) Show that the operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ obey

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{p})\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{p}), \tag{10}$$

with all other commutators between these operators vanishing. This implies that we have two sets of independent creation and annihilation operators for each $\mathbf{k}$.

The vacuum state $|0\rangle$ is defined to be the unique state where $a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. The states $a^\dagger(\mathbf{k})|0\rangle$ and $b^\dagger(\mathbf{k})|0\rangle$ each contain one particle, while the state $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle$ contains two particles, and so on.

b) In problem set 1, you showed that the complex scalar field Lagrangian had the symmetry $\Phi \to e^{i\alpha}\Phi$, $\Phi^* \to e^{-i\alpha}\Phi^*$. Compute the Noether current J^μ and conserved charge Q associated with the symmetry.

c) Write Q in terms of the creation and annihilation operators. You should find the result is indeterminate up to a constant; resolve this by defining the vacuum to have zero charge, $Q|0\rangle = 0$. What is the charge in the three other states mentioned above?

d) To check that this is the same symmetry operation that we started out with, we can see how it acts on the operators of the theory. Quantum mechanically, symmetries act on operators by conjugation, and we expect to have

$$e^{i\alpha Q}\, \Phi(\mathbf{x})\, e^{-i\alpha Q} = e^{i\alpha}\, \Phi(\mathbf{x}), \qquad e^{i\alpha Q}\, \Phi^*(\mathbf{x})\, e^{-i\alpha Q} = e^{-i\alpha}\, \Phi^*(\mathbf{x}). \tag{11}$$

Show that this implies the commutation relations

$$[Q, \Phi(\mathbf{x})] = \Phi(\mathbf{x}), \qquad [Q, \Phi^*(\mathbf{x})] = -\Phi^*(\mathbf{x}) \tag{12}$$

and show that these relations hold. Thus, Q generates phase rotations of the field.

e) Use Noether's theorem to find the stress-energy tensor $T^{\mu\nu}$ and the associated conserved total four-momentum P^μ.

f) Write the Hamiltonian $H = P^0$ and the spatial momenta $\mathbf{P}$ in terms of the creation and annihilation operators, again defining the vacuum to have zero four-momentum. What is the four-momentum in the three other states mentioned above?

g) By definition, the operator H generates time translations of the field – this is the content of the Schrodinger equation, $i\partial_t|\Psi\rangle = H|\Psi\rangle$, which holds unchanged in quantum field theory. As for the momenta $\mathbf{P}$, we expect

$$e^{i\mathbf{a}\cdot\mathbf{P}}\, \Phi(\mathbf{x})\, e^{-i\mathbf{a}\cdot\mathbf{P}} = \Phi(\mathbf{x} - \mathbf{a}) \tag{13}$$

for any vector $\mathbf{a}$, with a similar result for all the other fields. Show that this implies

$$[P^i, \Phi(\mathbf{x})] = -i\partial^i \Phi(\mathbf{x}) \tag{14}$$

and show that this relation holds. Thus, $\mathbf{P}$ generates spatial translations.

3. ⋆ Conserved currents of Lorentz transformations. (10 points)

This somewhat tricky problem combines the ideas of the first two. It is completely optional: the problem set will be graded out of 40 points, so that you will receive up to 100% credit if you don't do this problem, and up to 125% credit if you do.

Under a Lorentz transformation, a scalar field profile ϕ gets mapped to ϕ', so that $\phi'(x') = \phi(x)$. This implies that

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{15}$$

For an infinitesimal Lorentz transformation (2), this corresponds to

$$\phi'(x) = \phi(x) - \epsilon\, \omega^{\mu\nu} x_\nu \partial_\mu \phi(x) \tag{16}$$

to first order in ϵ. Because an infinitesimal Lorentz transformation is parametrized by a rank 2 tensor $\omega^{\mu\nu}$, the corresponding Noether current will be a rank 3 tensor $J^{\mu\nu\rho}$, where the first index is the usual index that comes from Noether's theorem, and the last two describe the Lorentz transformation. For simplicity, you can do the entire problem for a real scalar field. (This corresponds to a complex scalar field with $a(\mathbf{k}) = b(\mathbf{k})$, which in turn implies $\Phi = \Phi^*$ and $\Pi = \Pi^*$.)

a) Show that for a scalar field,

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{17}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Let the associated conserved charges be $M^{\mu\nu}$.

b) Since $M^{\mu\nu}$ is antisymmetric, there are six independent conserved charges. The three independent M^{ij} physically correspond to the angular momentum of the field. What is the physical meaning of the other three conserved quantities M^{0i}?

c) Show that after normal ordering, the angular momentum is

$$M^{ij} = i \int \frac{d^3k}{(2\pi)^3}\, a(\mathbf{k})^\dagger \left(k^j \frac{\partial}{\partial k_i} - k^i \frac{\partial}{\partial k_j} \right) a(\mathbf{k}) \tag{18}$$

The form of this answer implies that the particles created and annihilated by scalar fields do not carry any intrinsic angular momentum (i.e. spin).