PHYS-330: QFT I — Solutions 2

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1. Lorentz transformations. (10 points)

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ are linear transformations that leave inner products invariant, meaning that $x^\mu y_\mu = x'^\mu y'_\mu$ for any four-vectors x and y.

a) Show that this implies

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu. \tag{1}$$

Solution: Since the inner product stays the same, we have

$$\eta_{\mu\nu} x^\mu y^\nu = \eta_{\rho\sigma} x'^\rho y'^\sigma = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu x^\mu y^\nu. \tag{S1}$$

In other words, we have

$$(\eta_{\mu\nu} - \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu) x^\mu y^\nu = 0. \tag{S2}$$

for any four-vectors x and y, which implies the quantity in parentheses vanishes. (Concretely, we could take $x^\mu = \delta^\mu_0$ and $y^\nu = \delta^\nu_0$ to show the $\mu = \nu = 0$ component vanishes, and so on for all 16 pairs of values.)

b) All proper, orthochronous Lorentz transformations (i.e. all those which preserve the orientation of space and the direction of time) can be decomposed into infinitesimal Lorentz transformations. These take the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon\, \omega^\mu{}_\nu x^\nu \tag{2}$$

where ϵ is infinitesimal. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Solution: Plugging $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\, \omega^\mu{}_\nu$ into Eq. (1) yields

$$\eta_{\mu\nu} = \eta_{\rho\sigma}(\delta^\rho_\mu \delta^\sigma_\nu + \epsilon\, \omega^\rho{}_\mu \delta^\sigma_\nu + \epsilon\, \delta^\rho_\mu \omega^\sigma{}_\nu) \tag{S3}$$

where we dropped a term of order ϵ^2. Contracting the indices on the right-hand side,

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, \omega^\rho{}_\mu \eta_{\rho\nu} + \epsilon\, \eta_{\mu\sigma}\omega^\sigma{}_\nu \tag{S4}$$

from which we conclude $\omega_{\nu\mu} + \omega_{\mu\nu} = 0$, as desired.

c) The elements of any infinitesimal Lorentz transformation $\omega^\mu{}_\nu$ can be written as a 4×4 matrix, where μ and ν index the row and column, respectively. For an infinitesimal rotation $\epsilon = d\theta$ about the z-axis, write out this matrix, and denote it by iJ^3 for later. What exponential of J^3 corresponds to a finite rotation by an angle θ?

Solution: An infinitesimal rotation keeps t and z the same, and maps $x \to x + \epsilon y$ and $y \to y - \epsilon x$, so we can immediately write down

$$iJ^3 = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix}. \tag{S5}$$

The appropriate matrix exponential is $e^{i\theta J^3}$. The purpose of the factor of i in the definition is to ensure J^3 is a Hermitian matrix, in accordance with how it's treated in quantum mechanics.

d) Write down the matrix iK^1 corresponding to an infinitesimal boost by $\epsilon = dv$ about the x-axis. What exponential of K^1 corresponds to a finite boost by a velocity v?

Solution: An infinitesimal boost keeps y and z the same, and maps $x \to x + \epsilon t$ and $t \to t + \epsilon x$, so

$$iK^1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix}. \tag{S6}$$

Note that this is *not* an antisymmetric matrix. The quantity that is antisymmetric is $\omega_{\mu\nu}$, with two indices down, but raising an index to get $\omega^{\mu}{}_{\nu}$ produces some signs. The appropriate exponential for a finite boost is $e^{i\alpha K^1}$, where $\alpha = \tanh^{-1} v$ is the rapidity. (The most common mistake was instead writing e^{ivK^1}.)

There's a minor subtlety: the usual taught form of the Lorentz transformation has $x' = \gamma(x - vt)$, which suggests the matrix should have minus signs. But in that form, we're keeping the situation the same, and x' denotes the coordinate seen by an observer moving to the right. In this problem, we're considering the case where there's just one observer but physical objects are boosted to the right. Nonetheless, if you used the "passive" convention instead, it's no big deal since none of the other results in this problem will be affected.

Note that the factor of i in the definition of K^1 is a common convention, but it implies K^1 is *not* Hermitian. This is also sensible, as K^1 doesn't actually correspond to a physical observable. At a deeper level, it is related to the fact that non-compact non-Abelian Lie groups don't have finite-dimensional unitary representations, which will be discussed in section.

e) Defining J^1, J^2, K^2, and K^3 similarly, the generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{3}$$

This is the Lie algebra of the Lorentz group. Physically, these results tells us that infinitesimal rotations and boosts are vectors (i.e. angular velocity and velocity are vectors), and that composing boosts can yield a rotation. Prove these results for the cases $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$. (The proofs for other cases are similar.)

Solution: The full set of generators is, up to a factor of i,

$$iJ^1 = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \qquad iJ^2 = \begin{pmatrix} & & & \\ & & & -1 \\ & & & \\ & 1 & & \end{pmatrix} \qquad iJ^3 = \begin{pmatrix} & & & \\ & & 1 & \\ & -1 & & \\ & & & \end{pmatrix} \tag{S7}$$

$$iK^1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & \\ & & & \end{pmatrix} \qquad iK^2 = \begin{pmatrix} & & 1 & \\ & & & \\ 1 & & & \\ & & & \end{pmatrix} \qquad iK^3 = \begin{pmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \tag{S8}$$

For the case $(i, j) = (1, 1)$, the first and third relations are clearly satisfied, because the right-hand side is just zero, and $[J^1, K^1] = 0$ because these matrices are nonzero only in independent blocks. For the case $(i, j) = (1, 2)$, we need to check $[J^1, J^2] = iJ^3$, $[J^1, K^2] = iK^3$, and $[K^1, K^2] = -iJ^3$, which is straightforward to do manually.

2. Quantization of the complex scalar field. (30 points)

This will be an involved problem, but it will teach you everything there is to know about free field mode expansions. (You don't have to write every detail; when it's clear other computations will go the same way as one you just did, it's fine to just say so and move on.) In problem set 1 we considered a complex scalar field Φ with Lagrangian density

$$\mathcal{L} = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi, \tag{4}$$

and found the canonical fields Φ and Φ^* and momenta Π and Π^*. Letting $\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}$, introduce the following mode expansion for the canonical fields and momenta:

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{5}$$

$$\Phi^*(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{6}$$

$$\Pi(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{7}$$

$$\Pi^*(\mathbf{x}) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{8}$$

We quantize the fields by imposing the canonical commutation relations

$$[\Phi(\mathbf{x}), \Pi(\mathbf{y})] = [\Phi^*(\mathbf{x}), \Pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{9}$$

with all other commutators between these fields vanishing. (Technically, we should write Hermitian conjugates $\Phi^\dagger$ and $\Pi^\dagger$ here, since Φ and Π are now operators, but we'll continue to use stars to emphasize the links between the classical and quantum theories.)

a) Show that the operators $a(\mathbf{k})$, $a^\dagger(\mathbf{k})$, $b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ obey

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{p})\right] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{p}), \tag{10}$$

with all other commutators between these operators vanishing. This implies that we have two sets of independent creation and annihilation operators for each $\mathbf{k}$.

Solution: In order to keep the expressions as simple as possible, we'll define

$$d\mathbf{x} = d^3x, \qquad đ\mathbf{p} = \frac{d^3p}{(2\pi)^3}, \qquad \not{\delta}(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p}). \tag{S9}$$

As a simple of this notation, we have

$$\int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}} = \not{\delta}(\mathbf{p}), \qquad \int đ\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} = \delta(\mathbf{x}). \tag{S10}$$

If you just remember that every momentum delta function comes with a slash, and every momentum integral comes with a bar, all the factors of 2π will take care of themselves.

Now, we note that

$$\int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\Phi(\mathbf{x}) = \int \frac{d\mathbf{x}\, đ\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k})e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{x}}\right] \tag{S11}$$

$$= \int \frac{đ\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k})\not{\delta}(\mathbf{k}+\mathbf{k}') + b^\dagger(\mathbf{k})\not{\delta}(\mathbf{k}'-\mathbf{k})\right] \tag{S12}$$

$$= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}(a(-\mathbf{k}) + b^\dagger(\mathbf{k})). \tag{S13}$$

By a similar argument, we have

$$\int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}(i\Pi(\mathbf{x})) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}(b(-\mathbf{k}) - a^\dagger(\mathbf{k})). \tag{S14}$$

Combining these equations to solve for the creation and annihilation operators yields

$$a(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x})\right], \tag{S15}$$

$$b(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) + i\Pi(\mathbf{x})\right], \tag{S16}$$

$$a^\dagger(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) - i\Pi(\mathbf{x})\right], \tag{S17}$$

$$b^\dagger(\mathbf{k}) = \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) - i\Pi^*(\mathbf{x})\right]. \tag{S18}$$

Let's start by considering commutators involving $a(\mathbf{k})$. We only have nonzero commutators between Φ and Π, and between Φ^* and Π^*. This automatically implies that $[a(\mathbf{k}), a(\mathbf{p})] = [a(\mathbf{k}), b^\dagger(\mathbf{p})] = 0$, leaving just two to manually check. First, we have

$$\begin{aligned}
[a(\mathbf{k}), b(\mathbf{p})] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} [\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) + i\Pi(\mathbf{y})] && \text{(S19)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{y}} (\delta(\mathbf{y}-\mathbf{x})(\omega_{\mathbf{p}} - \omega_{\mathbf{k}})) && \text{(S20)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} (\omega_{\mathbf{p}} - \omega_{\mathbf{k}}) && \text{(S21)}\\
&= \not{\delta}(\mathbf{k}+\mathbf{p})\, \frac{\omega_{\mathbf{p}} - \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = 0 && \text{(S22)}
\end{aligned}$$

where the final step follows from $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$. Next, we have

$$\begin{aligned}
[a(\mathbf{k}), a^\dagger(\mathbf{p})] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} [\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) - i\Pi(\mathbf{y})] && \text{(S23)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{y}} (\delta(\mathbf{y}-\mathbf{x})(\omega_{\mathbf{p}} + \omega_{\mathbf{k}})) && \text{(S24)}\\
&= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} (\omega_{\mathbf{p}} + \omega_{\mathbf{k}}) && \text{(S25)}\\
&= \not{\delta}(\mathbf{k}-\mathbf{p})\, \frac{\omega_{\mathbf{p}} + \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = \not{\delta}(\mathbf{k}-\mathbf{p}) && \text{(S26)}
\end{aligned}$$

as desired. All of the other commutators can be derived similarly.

The vacuum state $|0\rangle$ is defined to be the unique state where $a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. The states $a^\dagger(\mathbf{k})|0\rangle$ and $b^\dagger(\mathbf{k})|0\rangle$ each contain one particle, while the state $a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle$ contains two particles, and so on.

b) In problem set 1, you showed that the complex scalar field Lagrangian had the symmetry $\Phi \to e^{i\alpha}\Phi$, $\Phi^* \to e^{-i\alpha}\Phi^*$. Compute the Noether current J^μ and conserved charge Q associated with the symmetry.

Solution: The infinitesimal transformation is $\delta\Phi = i\Phi$ and $\delta\Phi^* = -i\Phi$, so Noether's theorem gives

$$\begin{aligned}
J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\,\delta\Phi^* && \text{(S27)}\\
&= i((\partial^\mu\Phi^*)\Phi - (\partial^\mu\Phi)\Phi^*). && \text{(S28)}
\end{aligned}$$

The charge is just the integral of the charge density,

$$Q = i\int d\mathbf{x}\,((\partial^0\Phi^*)\Phi - (\partial^0\Phi)\Phi^*). \tag{S29}$$

Of course, if we want to evaluate this in the canonically quantized theory, we should express it in terms of the fields and canonical momenta, giving the final answer

$$Q = i\int d\mathbf{x}\,(\Pi(\mathbf{x})\Phi(\mathbf{x}) - \Pi^*(\mathbf{x})\Phi^*(\mathbf{x})). \tag{S30}$$

c) Write Q in terms of the creation and annihilation operators. You should find the result is indeterminate up to a constant; resolve this by defining the vacuum to have zero charge, $Q|0\rangle = 0$. What is the charge in the three other states mentioned above?

Solution: The fundamental reason the answer will be undetermined up to a constant is that Noether's theorem, which applies to classical fields, tells us (S30), but not the order that $\Pi(\mathbf{x})$ and $\Phi(\mathbf{x})$ are multiplied together. In the classical theory, that simply doesn't matter, but in the quantum theory it does because the fields are noncommuting operators. So without further information, the "right" answer in the quantum theory could contain $\Pi(\mathbf{x})\Phi(\mathbf{x})$, or $\Phi(\mathbf{x})\Pi(\mathbf{x})$, or even a mixture like $(\Pi(\mathbf{x})\Phi(\mathbf{x})+\Phi(\mathbf{x})\Pi(\mathbf{x}))/2$.

This is called an ordering ambiguity, and in general there is no way to resolve it. (Ordering information is intrinsically quantum; it simply doesn't exist in the classical theory!) But in simple cases like this, all of the possible orderings give you the same operator Q, just shifted by a constant. So we can fix the ambiguity by demanding the reasonable assumption $Q|0\rangle = 0$, which is known as normal ordering.

Anyway, let's not worry about this for now, and just directly evaluate the result. First,

$$\int d\mathbf{x}\, i\Pi(\mathbf{x})\Phi(\mathbf{x}) = \int \frac{d\mathbf{x}\, \bar{d}\mathbf{k}\, \bar{d}\mathbf{k}'}{2} \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}\, (b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S31}$$

$$\begin{aligned} &= \int \frac{\bar{d}\mathbf{k}\, \bar{d}\mathbf{k}'}{2} \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} \big[(b(\mathbf{k})a(\mathbf{k}') - a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'))\bar{\delta}(\mathbf{k}+\mathbf{k}') \\ &\qquad + (b(\mathbf{k})b^\dagger(\mathbf{k}') - a^\dagger(\mathbf{k})a(\mathbf{k}'))\bar{\delta}(\mathbf{k}-\mathbf{k}')\big] \end{aligned} \tag{S32}$$

$$= \int \frac{\bar{d}\mathbf{k}}{2}\, (b(\mathbf{k})a(-\mathbf{k}) - a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})). \tag{S33}$$

By a very similar computation, the second term is

$$\int d\mathbf{x}\, (-i\Pi^*(\mathbf{x}))\Phi^*(\mathbf{x}) = \int \frac{\bar{d}\mathbf{k}}{2}\, (-a(\mathbf{k})b(-\mathbf{k}) + b(\mathbf{k})^\dagger a(-\mathbf{k})^\dagger - a(\mathbf{k})a(\mathbf{k})^\dagger + b(\mathbf{k})^\dagger b(\mathbf{k})). \tag{S34}$$

The first term of (S33) cancels with the first term of (S34). To see this very explicitly:

$$\int \bar{d}\mathbf{k}\, b(\mathbf{k})a(-\mathbf{k}) = \int \bar{d}\mathbf{k}'\, a(\mathbf{k}')b(-\mathbf{k}') = \int \bar{d}\mathbf{k}\, a(\mathbf{k})b(-\mathbf{k}') \tag{S35}$$

where we changed the integration variable to $\mathbf{k}' = -\mathbf{k}$, used the commutation relation, and then renamed $\mathbf{k}'$ to $\mathbf{k}$. Similarly, the second term of (S33) cancels with the second term of (S34). This leaves

$$Q = \int \frac{\bar{d}\mathbf{k}}{2} (b(\mathbf{k})b^\dagger(\mathbf{k}) + b(\mathbf{k})^\dagger b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}) - a(\mathbf{k})a(\mathbf{k})^\dagger). \tag{S36}$$

At this point, we need to impose normal ordering. We will have $Q|0\rangle = 0$ if all the annihilation operators are on the right, so we move them to the right and throw out the constants this produces, giving

$$Q = \int \bar{d}\mathbf{k}\, (b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})). \tag{S37}$$

Evidently, the conserved charge is the number of type b particles minus the number of type a particles. So the charges in the three states mentioned above are -1, 1, and -2, respectively.

d) To check that this is the same symmetry operation that we started out with, we can see how it acts on the operators of the theory. Quantum mechanically, symmetries act on operators by conjugation, and we expect to have

$$e^{i\alpha Q}\, \Phi(\mathbf{x})\, e^{-i\alpha Q} = e^{i\alpha}\, \Phi(\mathbf{x}), \qquad e^{i\alpha Q}\, \Phi^*(\mathbf{x})\, e^{-i\alpha Q} = e^{-i\alpha}\, \Phi^*(\mathbf{x}). \tag{11}$$

Show that this implies the commutation relations

$$[Q, \Phi(\mathbf{x})] = \Phi(\mathbf{x}), \qquad [Q, \Phi^*(\mathbf{x})] = -\Phi^*(\mathbf{x}) \tag{12}$$

and show that these relations hold. Thus, Q generates phase rotations of the field.

Solution: Taking α infinitesimal, we have

$$(1 + i\alpha Q)\Phi(\mathbf{x})(1 - i\alpha Q) = (1 + i\alpha)\Phi(\mathbf{x}) \tag{S38}$$

and matching up the order α terms gives

$$iQ\Phi - i\Phi Q = i\Phi \tag{S39}$$

which is precisely the desired result. The proof for Φ^* is similar.

To show the first commutation relation, we have

$$\begin{aligned}
[Q, \Phi(\mathbf{x})] &= \int \frac{d\!\!{}^{-}\mathbf{k}\, d\!\!{}^{-}\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} [b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] && \text{(S40)}\\
&= \int \frac{d\!\!{}^{-}\mathbf{k}\, d\!\!{}^{-}\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left([b^\dagger(\mathbf{k})b(\mathbf{k}), b^\dagger(\mathbf{k}')]e^{-i\mathbf{k}'\cdot\mathbf{x}} - [a^\dagger(\mathbf{k})a(\mathbf{k}), a(\mathbf{k}')]e^{i\mathbf{k}'\cdot\mathbf{x}}\right) && \text{(S41)}\\
&= \int \frac{d\!\!{}^{-}\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\right) = \Phi(\mathbf{x}) && \text{(S42)}
\end{aligned}$$

as desired. The proof of the other commutation relation is similar, but we don't have to write it because it follows immediately from the result we just proved. Taking the Hermitian conjugate of both sides,

$$\Phi^*(\mathbf{x}) = [Q, \Phi(\mathbf{x})]^\dagger = (Q\Phi(\mathbf{x}) - \Phi(\mathbf{x})Q)^\dagger = \Phi^*(\mathbf{x})Q - Q\Phi^*(\mathbf{x}) = -[Q, \Phi^*(\mathbf{x})] \tag{S43}$$

which is the desired result. (Here we used the fact that Q is Hermitian.)

e) Use Noether's theorem to find the stress-energy tensor $T^{\mu\nu}$ and the associated conserved total four-momentum P^μ.

Solution: Using the same procedure as in class, and suppressing $\mathbf{x}$ arguments, we have

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\partial^\nu\Phi^* - \eta^{\mu\nu}\mathcal{L} && \text{(S44)}\\
&= \partial^\mu\Phi^*\,\partial^\nu\Phi + \partial^\mu\Phi\,\partial^\nu\Phi^* - \eta^{\mu\nu}\left((\partial_\rho\Phi^*)(\partial^\rho\Phi) - m^2\Phi^*\Phi\right). && \text{(S45)}
\end{aligned}$$

The total energy is

$$\begin{aligned}
P^0 &= \int d\mathbf{x}\, T^{00} && \text{(S46)}\\
&= \int d\mathbf{x}\, \partial^0\Phi\partial^0\Phi^* + \partial^0\Phi^*\partial^0\Phi - (\partial_\rho\Phi^*)(\partial^\rho\Phi) + m^2\Phi^*\Phi && \text{(S47)}\\
&= \int d\mathbf{x}\, \Pi^*\,\Pi + (\nabla\Phi^*)\cdot(\nabla\Phi) + m^2\Phi^*\Phi. && \text{(S48)}
\end{aligned}$$

The total momentum is

$$\begin{aligned}
P^i &= \int d\mathbf{x}\, T^{0i} && \text{(S49)}\\
&= \int d\mathbf{x}\, \partial^0\Phi^*\partial^i\Phi + \partial^0\Phi\partial^i\Phi^* && \text{(S50)}\\
&= \int d\mathbf{x}\, \Pi\,\partial^i\Phi + \Pi^*\,\partial^i\Phi^*. && \text{(S51)}
\end{aligned}$$

f) Write the Hamiltonian $H = P^0$ and the spatial momenta $\mathbf{P}$ in terms of the creation and annihilation operators, again defining the vacuum to have zero four-momentum. What is the four-momentum in the three other states mentioned above?

Solution: Starting with the Hamiltonian, we have three terms:

$$\int d\mathbf{x}\,\Pi^*\Pi = -\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \frac{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}{2} (a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(b(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} - a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S52}$$

$$\int d\mathbf{x}(\nabla\Phi^*)\cdot(\nabla\Phi) = -\int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} (\mathbf{k}b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - \mathbf{k}a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\cdot(\mathbf{k}'a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} - \mathbf{k}'b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S53}$$

$$\int d\mathbf{x}\, m^2\Phi^*\Phi = \int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} m^2 (b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}). \tag{S54}$$

When we do the $d\mathbf{x}$ integrations, each line gives four terms, along with appropriate delta functions,

$$\int d\mathbf{x}\,\Pi^*\Pi = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}\,\omega_{\mathbf{k}}^2(-a(\mathbf{k})b(-\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) - b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})) \tag{S55}$$

$$\int d\mathbf{x}\,(\nabla\Phi^*)\cdot(\nabla\Phi) = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}|\mathbf{k}|^2(b(\mathbf{k})a(-\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})) \tag{S56}$$

$$\int d\mathbf{x}\, m^2\Phi^*\Phi = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}\, m^2(b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})). \tag{S57}$$

Summing the terms, all the cross terms (those with one a and one b) cancel, and normal ordering leaves

$$H = \int \frac{đ\mathbf{k}}{2\omega_{\mathbf{k}}}(\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + m^2)(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \tag{S58}$$

$$= \int đ\mathbf{k}\,\omega_{\mathbf{k}}(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \tag{S59}$$

This tells us that each one of the a-type and b-type particles has energy $\omega_{\mathbf{k}}$. Thankfully, the momentum is a little easier to deal with. We have

$$\int d\mathbf{x}\,\Pi\,\partial^i\Phi = -i\int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\partial^i(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S60}$$

$$= \int \frac{d\mathbf{x} đ\mathbf{k} đ\mathbf{k}'}{2}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}}k'^i(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) \tag{S61}$$

$$= \int \frac{đ\mathbf{k}}{2}\,k^i(b(\mathbf{k})a(-\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})) \tag{S62}$$

The easy way to handle the other term in P^i is to notice that when you conjugate this term, you almost get the other term, but the operators are in the reverse order. But of course, we're going to fix the ordering ambiguity with normal ordering later, so this doesn't matter. So we can treat the other term as

$$\left(\int d\mathbf{x}\,\Pi\,\partial^i\Phi\right)^\dagger = \int \frac{đ\mathbf{k}}{2}\,k^i(a^\dagger(-\mathbf{k})b^\dagger(\mathbf{k}) + a(\mathbf{k})a^\dagger(\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + b(-\mathbf{k})a(\mathbf{k})) \tag{S63}$$

By substituting $\mathbf{k} = -\mathbf{k}'$ and then renaming $\mathbf{k}'$ back to $\mathbf{k}$, the first and last terms here cancel with the first and last terms of (S62). The second and third terms add, and normal ordering gives

$$P^i = \int đ\mathbf{k}\,k^i(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})) \tag{S64}$$

which tells us that a-type and b-type particles both carry momentum $\mathbf{k}$. Therefore, the states mentioned above have four-momentum $(\omega_{\mathbf{k}}, \mathbf{k})$, $(\omega_{\mathbf{k}}, \mathbf{k})$, and $(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \mathbf{k}_1 + \mathbf{k}_2)$.

g) By definition, the operator H generates time translations of the field – this is the content of the Schrodinger equation, $i\partial_t|\Psi\rangle = H|\Psi\rangle$, which holds unchanged in quantum field theory. As for the momenta $\mathbf{P}$, we expect

$$e^{i\mathbf{a}\cdot\mathbf{P}}\,\Phi(\mathbf{x})\,e^{-i\mathbf{a}\cdot\mathbf{P}} = \Phi(\mathbf{x} - \mathbf{a}) \tag{13}$$

for any vector $\mathbf{a}$, with a similar result for all the other fields. Show that this implies

$$[P^i, \Phi(\mathbf{x})] = -i\partial^i\Phi(\mathbf{x}) \tag{14}$$

and show that this relation holds. Thus, $\mathbf{P}$ generates spatial translations.

Solution: We again take infinitesimal $\mathbf{a}$ and expand, giving

$$(1 + i\mathbf{a} \cdot \mathbf{P})\Phi(\mathbf{x})(1 - i\mathbf{a} \cdot \mathbf{P}) = \Phi(\mathbf{x}) - \mathbf{a} \cdot \nabla\Phi(\mathbf{x}). \tag{S65}$$

Equating the order $\mathbf{a}$ components and taking care to remember that $\partial^i = -\partial_i$ gives the desired result.

By the way, you might wonder why we're treating the Hamiltonian and momentum slightly differently. The reason is that in this problem set, we're still working in Schrodinger picture, where the operators $\Phi(\mathbf{x})$ depend on space but not time. In Heisenberg picture, the Heisenberg equation of motion for operators gives the analogous result $[H, \Phi(\mathbf{x}, t)] = -i\partial^0\Phi(\mathbf{x}, t)$, so that in general we have $[P^\mu, \Phi(x)] = -i\partial^\mu\Phi(x)$.

Now, evaluating the commutator, we have

$$[P^i, \Phi(\mathbf{x})] = \int đ\mathbf{k}đ\mathbf{k}' \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}}[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] \tag{S66}$$

$$= \int đ\mathbf{k}đ\mathbf{k}' \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}}[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}), a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}] \tag{S67}$$

$$= \int đ\mathbf{k} \, \frac{k^i}{\sqrt{2\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}) \tag{S68}$$

$$= -i\partial^i \int \frac{đ\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}) = -i\partial^i\Phi(\mathbf{x}). \tag{S69}$$

3. ⋆ Conserved currents of Lorentz transformations. (10 points)

This somewhat tricky problem combines the ideas of the first two. It is completely optional: the problem set will be graded out of 40 points, so that you will receive up to 100% credit if you don't do this problem, and up to 125% credit if you do.

Under a Lorentz transformation, a scalar field profile ϕ gets mapped to ϕ', so that $\phi'(x') = \phi(x)$. This implies that

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{15}$$

For an infinitesimal Lorentz transformation (2), this corresponds to

$$\phi'(x) = \phi(x) - \epsilon\,\omega^{\mu\nu}x_\nu\partial_\mu\phi(x) \tag{16}$$

to first order in ϵ. Because an infinitesimal Lorentz transformation is parametrized by a rank 2 tensor $\omega^{\mu\nu}$, the corresponding Noether current will be a rank 3 tensor $J^{\mu\nu\rho}$, where the first index is the usual index that comes from Noether's theorem, and the last two describe the Lorentz transformation. For simplicity, you can do the entire problem for a real scalar field. (This corresponds to a complex scalar field with $a(\mathbf{k}) = b(\mathbf{k})$, which in turn implies $\Phi = \Phi^*$ and $\Pi = \Pi^*$.)

a) Show that for a scalar field,

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} \tag{17}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Let the associated conserved charges be $M^{\mu\nu}$.

Solution: This is just like the derivation of the stress-energy tensor, except that the constant displacement a^μ is replaced with $\omega^{\mu\nu}x_\nu$. Thus, the current for a given ω is

$$J^\mu = \omega^{\rho\nu}x_\nu T^\mu{}_\rho = \frac{1}{2}\omega_{\rho\nu}(T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho) \tag{S70}$$

where we used the antisymmetry of ω in the second step. The general rank-three conserved tensor can be found by "stripping off" the $\omega_{\rho\nu}$, giving

$$J^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho \tag{S71}$$

as desired.

b) Since $M^{\mu\nu}$ is antisymmetric, there are six independent conserved charges. The three independent M^{ij} physically correspond to the angular momentum of the field. What is the physical meaning of the other three conserved quantities M^{0i}?

Solution: We have

$$M^{0i} = \int d\mathbf{x}\, J^{00i} = \int d\mathbf{x}\, T^{0i}x^0 - T^{00}x^i. \tag{S72}$$

The first term is just the total momentum P^i times the time t. The second term is an integral of the energy density times x^i, so it is proportional to the location of the "center of energy",

$$X^i_{\mathrm{CE}} = \frac{\int d\mathbf{x}\, T^{00}x^i}{\int d\mathbf{x}\, T^{00}} = \frac{\int d\mathbf{x}\, T^{00}x^i}{E} \tag{S73}$$

where E is the total energy. Plugging in these definitions, we have

$$M^{0i} = P^i t - E X^i_{\mathrm{CE}}. \tag{S74}$$

Since this is conserved, its time derivative vanishes, giving

$$\frac{P^i}{E} = \frac{dX^i_{\mathrm{CE}}}{dt}. \tag{S75}$$

In other words, the conservation of M^{0i} means that the center of energy moves at constant speed. This is a very general principle in relativistic theories.

c) Show that after normal ordering, the angular momentum is

$$M^{ij} = i\int \frac{d^3k}{(2\pi)^3}\, a(\mathbf{k})^\dagger \left(k^j \frac{\partial}{\partial k_i} - k^i \frac{\partial}{\partial k_j}\right) a(\mathbf{k}) \tag{18}$$

The form of this answer implies that the particles created and annihilated by scalar fields do not carry any intrinsic angular momentum (i.e. spin).

Solution: We have

$$M^{ij} = \int d\mathbf{x}\, J^{0ij} = \int d\mathbf{x}\, x^i T^{0j} - x^j T^{0i}. \tag{S76}$$

The resulting integrals look very similar to those we already did in 2(f), but the factor of x^i complicates things: it means we can no longer do the $d\mathbf{x}$ integral trivially. The trick is to convert the factor of x^i into a momentum derivative. From a starting point much like (S60), we have

$$\begin{aligned}
\int d\mathbf{x}\, x^i \pi\, \partial^j \phi &= -\frac{i}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i (a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})\partial^j(a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S77)}\\
&= \frac{1}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i k'^j (a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}})(-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S78)}\\
&= -\frac{i}{2}\int d\mathbf{x} đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j (a(\mathbf{k})\partial_{k_i} e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})\partial_{k_i} e^{-i\mathbf{k}\cdot\mathbf{x}}) \\
&\qquad \times (-a(\mathbf{k}')e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger(\mathbf{k}')e^{-i\mathbf{k}'\cdot\mathbf{x}}) && \text{(S79)}\\
&= -\frac{i}{2}\int đ\mathbf{k} đ\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j \big((a(\mathbf{k})a(\mathbf{k}') - a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}'))\partial_{k_i} \delta\!\!\!/(\mathbf{k}+\mathbf{k}') \\
&\qquad + (a(\mathbf{k})a^\dagger(\mathbf{k}') - a^\dagger(\mathbf{k})a(\mathbf{k}'))\partial_{k_i} \delta\!\!\!/(\mathbf{k}-\mathbf{k}')\big). && \text{(S80)}
\end{aligned}$$

To evaluate the delta function, we need to integrate by parts, which causes the ∂_{k_i}'s to hit the rest of the integrand. Now, the ∂_{k_i} can either act on $\sqrt{\omega_\mathbf{k}}$ or the ladder operators. If it hits the $\sqrt{\omega_\mathbf{k}}$, it produces a factor of k^i which makes the integrand proportional to $k^i k'^j$. However, since M^{ij} involves an antisymmetrization in i and j, these terms don't contribute to the final answer. We only care about the terms where ∂_{k_i} acts on the ladder operators, giving

$$M^{ij} = \frac{i}{2}\int d\!\!{}^-\mathbf{k}\, k^j(-(\partial_{k_i}a(\mathbf{k}))a(-\mathbf{k}) + (\partial_{k_i}a^\dagger(\mathbf{k}))a^\dagger(-\mathbf{k}) + (\partial_{k_i}a(\mathbf{k}))a^\dagger(\mathbf{k}) - (\partial_{k_i}a^\dagger(\mathbf{k}))a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}). \tag{S81}$$

Let's first show that the first term vanishes. For convenience we define

$$I^{ij} = \int d\mathbf{k}\, k^j a(-\mathbf{k})\partial_{k_i}a(\mathbf{k}). \tag{S82}$$

Integrating by parts again, we have

$$\begin{aligned} I^{ij} &= -\int d\mathbf{k}\,\partial_{k_i}(k^j a(-\mathbf{k}))a(\mathbf{k}) && \text{(S83)}\\ &= -\int d\mathbf{k}\,\delta^{ij}a(-\mathbf{k})a(\mathbf{k}) + k^j(\partial_{k_i}a(-\mathbf{k}))a(\mathbf{k}) && \text{(S84)}\\ &= -\int d\mathbf{k}\,\delta^{ij}a(-\mathbf{k})a(\mathbf{k}) + k^j(\partial_{k_i}a(\mathbf{k}))a(-\mathbf{k}) && \text{(S85)}\\ &= -I^{ij} - \int d\mathbf{k}\,\delta^{ij}a(-\mathbf{k})a(\mathbf{k}) && \text{(S86)} \end{aligned}$$

where we did the usual $\mathbf{k} \to -\mathbf{k}$ trick in the third step. This shows that I^{ij} is proportional to δ^{ij}. Since the final answer M^{ij} involves an antisymmetrization in i and j, this shows the first term does not contribute to M^{ij}. Similarly, the second term doesn't contribute either.

You might be worried about what happens to the boundary terms when we integrate by parts. Usually, we justify throwing away boundary terms by saying the integrand goes to zero at infinity. But here, it doesn't: the integrand is always a nonzero operator, for any $\mathbf{k}$. However, as the integrand goes to infinity, we get creation and annihilation operators for extremely high momentum particles. These operators have zero matrix elements if you sandwich them between any two physically accessible states, so they might as well just be zero.

We are now left with the last two terms in (S81). Normal ordering now gives

$$\begin{aligned} M^{ij} &= \frac{i}{2}\int d\!\!{}^-\mathbf{k}\, k^j(a^\dagger(\mathbf{k})\partial_{k_i}a(\mathbf{k}) - (\partial_{k_i}a^\dagger(\mathbf{k}))a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}) && \text{(S87)}\\ &= \frac{i}{2}\int d\!\!{}^-\mathbf{k}\, k^j(a^\dagger(\mathbf{k})\partial_{k_i}a(\mathbf{k}) + a^\dagger(\mathbf{k})\partial_{k_i}a(\mathbf{k})) - (i \leftrightarrow j\,\text{term}) && \text{(S88)}\\ &= i\int d\!\!{}^-\mathbf{k}\, a^\dagger(\mathbf{k})(k^j\partial_{k_i} - k^i\partial_{k_j})a(\mathbf{k}) && \text{(S89)} \end{aligned}$$

where we integrated by parts again. This is the final result.

The only reason this result looks more complicated than our result for the spatial momentum is that we expanded the field in plane wave modes, which are eigenfunctions of translations but not rotations. Alternatively, we could have expanded the field in spherical waves (i.e. equal to a spherical Bessel function in r times a spherical harmonic), in which case the angular momentum expansion looks simple and the momentum expansion looks complicated. Plane waves are useful for describing the initial states in a particle physics experiment, but spherical waves can be useful in other situations, such as the emission of photons from a excited atom.