PHYS-330: QFT I — Problem Set 3 (due 10/18)

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1. Locality in relativistic quantum field theory. (20 points)

The vacuum two-point correlation function of a real scalar field is

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle. \tag{1}$$

It only depends on the difference of positions $z = x - y$, by translational invariance, and it quantifies correlations between the field values at x and y in the vacuum state.

a) Show that

$$D(z) = \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\delta(p^2-m^2)\,\theta(p^0)\,e^{-ip\cdot z} \tag{2}$$

where the Heaviside step function θ is 1 if the argument is positive and 0 otherwise.

b) Show that if $z'^\mu = \Lambda^\mu{}_\nu z^\nu$ for a proper orthochronous Lorentz transformation Λ, then $D(z') = D(z)$. (Hint: show that d^4p and $\delta(p^2-m^2)\,\theta(p^0)$ are each Lorentz invariant.)

c) Show that for spacelike separation, $z^2 < 0$, the correlation function is exponentially decaying but nonzero. You can do this in two ways. First, you can show that

$$D(z) = \frac{m}{4\pi^2\sqrt{-z^2}}\,K_1(\sqrt{-z^2}\,m) \tag{3}$$

where K_1 is a modified Bessel function of the second kind. Alternatively, you may numerically integrate $D(z)$ and graph the result. In both cases you may use any books or computer programs needed, such as Mathematica or Abramowitz and Stegun.

A key requirement for a relativistic theory is that it is local, meaning that effects don't propagate faster than the speed of light. That means any change applied to the field at x should only affect observable results at y if x and y aren't spacelike separated, $(y-x)^2 \geq 0$.

You might thus be concerned about part (c), which implies a field in the vacuum "knows" about the values of the field at spacelike separation. But there's nothing wrong with this. For example, to define a reference frame in special relativity, one synchronizes the clocks by sending light pulses throughout all space. After synchronization, all the clocks "know" about the values on the other clocks, even at spacelike separation. But that doesn't mean that changes propagate faster than light; it's just a consequence of how we set up the system. Similarly, ensuring a field is in the vacuum state requires absorbing all particles throughout all space, and this process sets up correlations between field values.

To test if our theory is local, we must see whether *changes* in the state propagate faster than light; that is the subject of the rest of the question.

d) One way to interact with a quantum field is to measure its value at a point, a process which changes the state. From quantum mechanics, we know that measurements of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not affect each other if $[\mathcal{O}_1, \mathcal{O}_2] = 0$. Show that

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = 0 \tag{4}$$

for spacelike separation, $(x-y)^2 < 0$, as expected from locality.

e) Another way to interact with a field is to couple it to a classical source $J(x)$, which for a real scalar field corresponds to taking the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{5}$$

This is the scalar analogue of driving the electromagnetic field with a current $J^\mu(x)$. For simplicity, we'll treat $\phi(x)$ as a classical field for now, though similar conclusions will hold when it is a quantum field. Show that the classical equation of motion is

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = J(x). \tag{6}$$

f) Show that the equation of motion is solved by

$$\phi(x) = i\int d^4y\, G(x-y)J(y) \tag{7}$$

where G is a Green's function of the Klein–Gordon operator, which means

$$(\partial_\mu \partial^\mu + m^2)\,G(z) = -i\,\delta^{(4)}(z). \tag{8}$$

The factors of $-i$ here are purely conventional, and will simplify results later.

g) By taking Fourier transforms, we may heuristically write the Green's function as

$$G(z) = \int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{p^2 - m^2} \tag{9}$$

which formally obeys Eq. (8). However, this expression is not mathematically well-defined because the integral blows up when $p^2 = m^2$. To get a definite result, we must add "$i\epsilon$" terms to the denominator to keep it from vanishing. There are multiple ways to do this, which physically corresponds to the fact that there are multiple possible Green's functions, depending on the field's boundary conditions. Three key examples are the retarded, advanced, and Feynman Green's functions,

$$G_R(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{(p^0+i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{10}$$

$$G_A(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{(p^0-i\epsilon)^2 - |\mathbf{p}|^2 - m^2} \tag{11}$$

$$G_F(z) = \lim_{\epsilon\to 0^+}\int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip\cdot z}}{p^2 - m^2 + i\epsilon} \tag{12}$$

where the limit notation means ϵ approaches zero from the positive end. By using the residue theorem to perform the integral over p^0, show that

$$G_R(z) = \theta(z^0)(D(z) - D(-z)) \tag{13}$$

$$G_A(z) = \theta(-z^0)(D(-z) - D(z)) \tag{14}$$

$$G_F(z) = \theta(z^0)D(z) + \theta(-z^0)D(-z) \tag{15}$$

h) The retarded Green's function applies when the field is zero before the source acts. Show that $G_R(x-y)$ vanishes for spacelike separation, as expected from locality.

i) The Feynman propagator will play a crucial role when we introduce Feynman diagrams because it is equal to $\langle 0|T\phi(x)\phi(y)|0\rangle$. Does it vanish for spacelike separation?

2. Recovering classical field theory. (10 points)

If an operator $\mathcal{O}$ is time-independent in Schrodinger picture, then in Heisenberg picture,

$$\frac{d\mathcal{O}(t)}{dt} = i\,[H(t), \mathcal{O}(t)]. \tag{16}$$

In quantum field theory, working in the Heisenberg picture allows fields to depend on spacetime, making results look more Lorentz invariant. For example, we saw that in Heisenberg picture, a real scalar field obeys the Klein–Gordan equation $(\partial^2 + m^2)\phi = 0$. Now consider the Lagrangian of Eq. (5), which additionally includes a source term $J(x)$. In this case, the Hamiltonian is explicitly time-dependent,

$$H(t) = H_0 - \int d^3\mathbf{x}\,\phi(\mathbf{x},t)J(\mathbf{x},t) \tag{17}$$

where H_0 is the free Hamiltonian.

a) Show that in Heisenberg picture, the field $\phi(x)$ obeys Eq. (6). (Thus, by the logic of problem 1, expectation values of a quantum field $\phi(x)$ respond locally to sources.)

With the source, quantum fields can evolve in time nontrivially. Suppose we start in the vacuum state $|0\rangle$, and at time $t = 0$ apply an impulse to the field via

$$J(x) = \delta(t)\,j(\mathbf{x}). \tag{18}$$

This is the scalar analogue of suddenly turning an electric current on and off in electromagnetism. It is also closely related to the last half of problem 2 of problem set 1.

b) The impulse causes operators to instantaneously shift in value at $t = 0$. Show that

$$\phi(\mathbf{x}, 0^+) = \phi(\mathbf{x}, 0^-), \qquad \pi(\mathbf{x}, 0^+) = \pi(\mathbf{x}, 0^-) + j(\mathbf{x}). \tag{19}$$

Here, 0^+ means a time right after $t = 0$, and 0^- means a time right before, i.e.

$$f(0^+) - f(0^-) = \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \frac{df(t)}{dt}\,dt \tag{20}$$

for any function of time.

c) Show that this is equivalent to the annihilation operators shifting by

$$a(\mathbf{p}, 0^+) = a(\mathbf{p}, 0^-) + \frac{i}{\sqrt{2\omega_{\mathbf{p}}}}\,\tilde{j}(\mathbf{p}) \tag{21}$$

where $\tilde{j}$ is the Fourier transform of j.

d) Show that the field is in a coherent state after the impulse, in the sense that the state is an eigenvector of $a(\mathbf{p}, 0^+)$. Then evaluate $\langle 0|a^\dagger(\mathbf{p}, 0^+)a(\mathbf{p}, 0^+)|0\rangle$, which gives the number of particles of momentum $\mathbf{p}$ produced, per unit volume of momentum space.

As you saw in problem set 1, coherent states of the quantum harmonic oscillator have the same position and momentum uncertainties as the vacuum state. Similarly, in quantum field theory, coherent states of the field have the same ϕ and π uncertainties as the vacuum, while their expectation values behave classically. When the driving is strong, the uncertainties become negligible compared to the expectation values, and the number of particles becomes large so that we can no longer see their discreteness. We therefore recover a classical field.

3. Practice with time-ordered exponentials. (10 points)

Suppose the Hamiltonian is $H = H_0 + H_{\text{int}}$, where H_0 is a time-independent free Hamiltonian, and H_{int} is an interaction which could be time-dependent. In interaction picture, operators evolve under the free Hamiltonian H_0 alone, and the states are $|\psi(t)\rangle_I$. Let's first review some results derived in lecture.

a) Show that
$$i\,\partial_t|\psi(t)\rangle_I = H_{\text{int},I}(t)|\psi(t)\rangle_I. \tag{22}$$
where $H_{\text{int},I}(t)$ is the interaction Hamiltonian in the interaction picture.

b) Show that if the interaction picture time evolution operator is defined as
$$|\psi(t_f)\rangle_I = U(t_f, t_i)|\psi(t_i)\rangle_I \tag{23}$$
for $t_f > t_i$, then it is given by Dyson's formula,
$$U(t_f, t_i) = T\exp\left(-i\int_{t_i}^{t_f} dt\, H_{\text{int},I}(t)\right) \tag{24}$$
where T denotes time ordering.

Now let's do some concrete calculations with the time evolution operator.

c) Suppose the interaction is only turned on for two moments, i.e. it has the form
$$H_{\text{int},I}(t) = g\,(h_1\,\delta(t-t_1) + h_2\,\delta(t-t_2)). \tag{25}$$
Write out $U(t_f, t_i)$ up to and including terms of order g^2, assuming $t_i < t_1 < t_2 < t_f$.

d) Now consider a general $H_{\text{int},I}(t)$ which is proportional to a coupling g. It is generally true that $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. Explicitly show that this result is true, up to and including terms of order g^2, in the case $t_1 < t_2 < t_3$.