PHYS-330: QFT I — Problem Set 4 (due 10/25)

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1. Using Wick's theorem. (7 points)

For both parts of this problem, you can use Wick's theorem for two real scalar fields.

a) Show that $T(\phi(x_1)\phi(x_2))$ and $:\phi(x_1)\phi(x_2):$ both remain the same when x_1 and x_2 are exchanged. Using these results, explain why $D_F(x_1 - x_2) = D_F(x_2 - x_1)$.

b) Prove Wick's theorem for a time ordered product of three scalar fields,

$$\begin{aligned} T(\phi(x_1)\phi(x_2)\phi(x_3)) = \; &:\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)D_F(x_2 - x_3) \\ &+ \phi(x_2)D_F(x_3 - x_1) + \phi(x_3)D_F(x_1 - x_2). \end{aligned} \tag{1}$$

2. Diagrams in ϕ^3 theory. (18 points)

Consider a real scalar field ϕ of mass m with a cubic self-interaction, so that

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \qquad \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\,\phi^3. \tag{2}$$

In this problem, you will draw some diagrams in this interacting theory and compute some symmetry factors. As in lecture, $|0\rangle$ is the free vacuum, the interaction Hamiltonian density is $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, and H_I is the interaction Hamiltonian in interaction picture.

a) Using Wick's theorem, evaluate the vacuum correlation function

$$\langle 0|\, T \exp\left(-i\int dt\, H_I(t)\right)|0\rangle \tag{3}$$

up to and including terms of order λ^2. Your result should be an explicit expression in terms of integrals of the position-space Feynman propagator. (You will run into terms proportional to $D_F(0)$, which is infinite. Don't worry about this for now; we'll come back to this issue when we discuss renormalization.)

b) Draw all distinct diagrams that contribute to the vacuum correlation function at order λ^3 and λ^4. For this part, you do not need to evaluate the diagrams or compute any symmetry factors, but you should neatly display all of your diagrams in one box.

c) Now consider the two-point correlation function

$$\langle\Omega|T\left(\phi_H(x)\phi_H(y)\right)|\Omega\rangle = \frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left(-i\int dt\, H_I\right)\right)|0\rangle}{\langle 0|T\exp\left(-i\int dt\, H_I\right)|0\rangle} \tag{4}$$

where ϕ_H and ϕ_I are Heisenberg and interaction picture fields. As discussed in class, only connected diagrams contribute to the left-hand side, i.e. diagrams where all fields are connected to at least one of the external points. Find all diagrams that contribute up to and including order λ^4, along with their symmetry factors, which you can compute with any method. Again, give your answer by drawing everything neatly inside one box. (Hint: there are 13 fully connected diagrams, i.e. diagrams where all fields are connected to *both* of the external points.)

3. Recovering nonrelativistic quantum mechanics. (15 points)

A free complex scalar field Φ of mass m satisfies the Klein–Gordan equation, so its plane wave modes $e^{-ip\cdot x}$ satisfy $\omega^2 = |\mathbf{k}|^2 + m^2$. This implies we have plane waves with both positive and negative ω, which is universal in relativistic theories. Upon quantization, excitations of these modes correspond to matter and antimatter, with opposite charges.

The nonrelativistic limit is $|\mathbf{k}| \ll m$, but since antimatter is inherently a feature of relativistic theories, we should also throw away the negative ω modes. To do this, take

$$\Phi(x) = e^{-imt}\,\chi(x)/\sqrt{2m} \tag{5}$$

and assume $|\dot{\chi}| \ll m\chi$, so that only nonrelativistic positive ω modes are excited. The $\sqrt{2m}$ factor transfers us back to nonrelativistic normalization.

a) Simplify the complex scalar field action to

$$S = \int d^4x \left(i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi \right) \tag{6}$$

by dropping a term that is small for $|\dot{\chi}| \ll m\chi$.

b) Find the Euler–Lagrange equations for χ and χ^*, and the conserved current $J^\mu = (\rho, \mathbf{J})$ corresponding to the symmetry $\chi \to e^{-i\alpha}\chi$ and $\chi^* \to e^{i\alpha}\chi^*$. (Be careful to account for minus signs from the relativistic metric.)

For the relativistic complex scalar field, we had to canonically quantize Φ, Π, Φ^*, and Π^* as two pairs of phase space variables. But for this nonrelativistic field, the canonical momentum for χ is just $i\chi^*$, so we only have one pair. This is because we threw away the negative ω solutions, and it implies the canonical commutation relations are

$$[\chi(\mathbf{x}), \chi^*(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{7}$$

with all other commutators vanishing. These can be satisfied with the mode expansion

$$\chi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \qquad \chi^*(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{8}$$

c) Show that $[a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$. (Of course, the other commutators vanish.) Then show that the single-particle states $|\mathbf{x}\rangle = \chi^*(\mathbf{x})|0\rangle$ and $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ behave like nonrelativistic position and momentum states, obeying

$$\langle \mathbf{y}|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{y} - \mathbf{x}), \quad \langle \mathbf{q}|\mathbf{p}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{q} - \mathbf{p}), \quad \langle \mathbf{p}|\mathbf{x}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{9}$$

The factors of 2π are just due to our Fourier transform convention.

d) A general single-particle state takes the form

$$|\psi(t)\rangle = \int d^3\mathbf{x}\, \psi(\mathbf{x}, t)|\mathbf{x}\rangle \tag{10}$$

where $\psi(\mathbf{x}, t)$ is the particle's position-space wavefunction at time t. As always in quantum theories, the state evolves over time as $i\,\partial_t|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian. Write this equation as a partial differential equation involving $\psi(\mathbf{x}, t)$.

e) Take the term in the action you dropped in part (a) and simplify it by applying the Euler–Lagrange equations you found in part (b). Find how the Hamiltonian in part (d) changes when this term is added. What is its physical interpretation?

4. ⋆ The force between sources. (5 points)

In this optional problem, we'll compute a directly measurable quantity with quantum field theory, for the first time in this course. So far we have only learned to compute correlation functions, which often do not have a direct physical interpretation. For example, the vacuum correlation function Eq. (3) can be used to find the energy of the interacting vacuum relative to the free vacuum, but this isn't directly measurable. (You can't turn off the interaction in real life.)

Instead, let's again consider a free real scalar field with source $J(x)$,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x). \tag{11}$$

If we change the vacuum by including a static source $J(x)$, the resulting shift in vacuum energy can be directly interpreted as the energy of the source, which is measurable.

a) Using Wick's theorem, show that

$$\langle 0| T \exp\left(i \int d^4x\, \phi(x)J(x)\right)|0\rangle = \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x-y)J(y)\right). \tag{12}$$

b) Now consider the case where two point sources are turned on for a long time T,

$$J(x) = g\, f(t)\,(\delta^{(3)}(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{R})), \qquad f(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}. \tag{13}$$

If you plug this into Eq. (12), you'll get infinite terms involving $D_F(0)$, reflecting the fact that point sources have infinite energy. Discarding these terms so we can focus on the sources' interaction energy, evaluate Eq. (12) assuming $T \gg R \gg 1/m$.

c) Relate your answer to the interaction energy $V(R)$ between the sources, and find it. Is the force between them attractive or repulsive?