PHYS-330: QFT I — Problem Set 5 (due 11/1)

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1. Mandelstam variables. (4 points)

Consider any $2 \to 2$ scattering process, where the two incoming particles have momenta p_1 and p_2, and the two outgoing particles have momenta p_3 and p_4. By momentum conservation, $p_1 + p_2 = p_3 + p_4$, and $p_i^2 = m_i^2$ where m_i is the mass of particle i. In this situation, it is often useful to work in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{1}$$

a) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

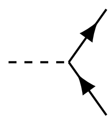
b) In the center of mass frame, the total energy is E_{cm} and the angle of $\mathbf{p}_1$ to $\mathbf{p}_3$ is θ. Write s, t, and u in terms of E_{cm} and θ, assuming all four particles are massless.

2. Scalar Yukawa amplitudes. (7 points)

Let ϕ be a real scalar field of mass M, and ψ be a complex scalar field of mass m, with

$$\mathcal{L}_{\text{int}} = -g\,\psi^* \psi \phi. \tag{2}$$

The field ψ can annihilate a particle, or create a particle with opposite $U(1)$ charge; these particles are conventionally called the ψ and ψ^*, respectively. Similarly, the field ϕ can create or annihilate a ϕ particle. (That is, particles are named after the field that annihilates them.) As discussed in section, the Feynman rules are

 $= \dfrac{i}{p^2 - M^2 + i\epsilon}$  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$  $= -ig$ (3)

where a dashed line stands for ϕ and a solid line stands for ψ.

a) Assuming $M > 2m$, calculate the decay rate of a ϕ particle to leading order in g. (Start from equation (4.86) of Peskin and Schroeder, and do the phase space integrals.)

b) Find the amplitude for $\psi(p_1)\psi^*(p_2) \to \psi(p_3)\psi^*(p_4)$ scattering to leading order in g, in terms of Mandelstam variables.

c) Suppose the energies we can reach in an experiment are higher than m, but much lower than M. In this case, we might not know that ϕ particles exist, since we can't produce them, so we would have to describe the scattering process in part (b) using an "effective" field theory in terms of ψ alone. Show that at leading order,

$$\mathcal{L}_{\text{int}} = -\lambda\,\psi^* \psi \psi^* \psi \tag{4}$$

will yield the same answer for part (b) for some value of λ, and find that value in terms of g and M. Assume all elements of the momenta p_i are much less than M.

3. Solving a trivial theory. (4 points)

Consider a free real scalar field of mass m, but treat the mass term as the perturbation,

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \qquad \mathcal{L}_{\text{int}} = -\frac{1}{2} m^2 \phi^2. \tag{5}$$

a) Write down the Feynman rules for this theory.

b) Evaluate the momentum-space propagator exactly, summing all of the (infinitely many) connected Feynman diagrams.

4. Pion scattering in the linear sigma model. (25 points)

Consider a theory with N free real scalar fields $\Phi_i(x)$ of equal mass m,

$$\mathcal{L}_0 = \sum_{i=1}^{N} \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i\Phi_i. \tag{6}$$

This Lagrangian is symmetric under rotations of the scalar fields among themselves. We can show this more clearly by defining an N-element vector $\mathbf{\Phi}$ with elements of Φ_i, so

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \mathbf{\Phi})^2 - \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi}. \tag{7}$$

This is just shorthand for Eq. (6). Note that the index on Φ_i is not a spatial index (such as on ∂_i). It is a "flavor" index, meaning it just identifies which field we're talking about.

a) We quantize the theory by imposing the equal time commutation relations

$$[\Pi_i(\mathbf{x},0), \Phi_j(\mathbf{y},0)] = i\delta_{ij}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad \Pi_i(x) = \partial_t \Phi_i(x), \tag{8}$$

with all other commutators vanishing. Show that the propagator is

$$\langle 0|T\{\Phi_i(x)\Phi_j(y)\}|0\rangle = \contraction{}{\Phi}{_i(x)}{\Phi}\Phi_i(x)\Phi_j(y) = \delta_{ij}\, D_F(x-y). \tag{9}$$

where δ_{ij} is 1 if $i=j$, and 0 otherwise. The momentum space Feynman rule is

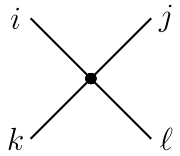
$$i \;\text{———}\; j \;=\; \frac{i\,\delta_{ij}}{p^2 - m^2 + i\epsilon} \tag{10}$$

where the i and j are flavor indices.

b) The linear sigma model additionally contains the interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{11}$$

which is also symmetric under rotations of $\mathbf{\Phi}$. Show that this interaction corresponds to the momentum space Feynman rule



$$= -2i\lambda\left(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right). \tag{12}$$

(Hint: consider the cases where all four fields have the same flavor, and when two pairs of fields have the same flavor.)

c) Find the *total* cross section in the centre of mass frame for the processes

$$\Phi_1\Phi_2 \to \Phi_1\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_2\Phi_2, \qquad \Phi_1\Phi_1 \to \Phi_1\Phi_1 \tag{13}$$

to leading order in λ, in terms of E_{cm}. (Hint: start from equation (4.85) of Peskin and Schroeder, and be careful with factors of 2.)

In the linear sigma model, the potential energy of a uniform classical field is

$$V(\Phi) = \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{\lambda}{4}\left(\mathbf{\Phi}\cdot\mathbf{\Phi}\right)^2. \tag{14}$$

When we quantize the harmonic oscillator, the usual definition of the creation and annihilation operators in terms of x and p only makes sense if the potential's minimum is at $x = 0$. If the minimum is somewhere else, then many things go wrong. For instance, $a|0\rangle$ won't be zero, and more generally $a^\dagger$ and a won't have simple commutation relations with H, so won't properly raise and lower the energy. Similarly, it only makes sense to quantize fields in the usual way about minima of the potential $V(\Phi)$.

If $m^2 > 0$ and $\lambda > 0$, there is a unique minimum of the potential at $\mathbf{\Phi} = \mathbf{0}$, so our treatment above makes sense. Now suppose that $m^2 < 0$ and $\lambda > 0$.

d) Defining $\mu^2 = -m^2$, show that the minima of the potential are at

$$|\mathbf{\Phi}| = v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{15}$$

The quantum theory thus has multiple vacuum states, but all have nonvanishing $\langle\mathbf{\Phi}\rangle$.

By symmetry under rotations of $\mathbf{\Phi}$, we can suppose without loss of generality that we are in a vacuum state where $\langle\Phi_N\rangle = v$, with all others vanishing. Define the new fields

$$\sigma(x) = \Phi_N(x) - v, \qquad \pi_i(x) = \Phi_i(x), \qquad i \in \{1, 2, \ldots, N-1\} \tag{16}$$

which all have vanishing expectation values in this vacuum.

e) Show that in terms of these fields, the Lagrangian is

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \\ & - \frac{\lambda_{4\pi}}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 - \lambda_{3\sigma}\sigma^3 - \frac{\lambda_{4\sigma}}{4}\sigma^4 - \lambda_{\pi\sigma 1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma - \lambda_{\pi\sigma 2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2 + C.\end{aligned} \tag{17}$$

and give expressions for m_σ, $\lambda_{4\pi}$, $\lambda_{3\sigma}$, $\lambda_{4\sigma}$, $\lambda_{\pi\sigma 1}$, $\lambda_{\pi\sigma 2}$ and C in terms of μ and λ. We see that σ is massive, but we have $N-1$ massless pion fields π_i.

f) Write the momentum-space Feynman rules in terms of μ and λ. Draw the pions with a solid line and flavor index i, and the σ particle with a double solid line. (Hint: if you've seen the pattern from the other Feynman rules in the problem set, it should be possible to write down the answer with some thought, but no explicit calculation. There is no Feynman rule corresponding to the constant C, which has no effect here.)

g) Compute the decay rate of a σ particle to leading order in λ, and give the corresponding lifetime in seconds if $\lambda = 0.1$, $\mu = 10$ GeV and $N = 3$.

h) Find the amplitude for $\pi_i(p_1)\pi_j(p_2) \to \pi_k(p_3)\pi_\ell(p_4)$ scattering to leading order in λ, in terms of Mandelstam variables. (Hint: there are four Feynman diagrams.)

i) Show that the amplitude in part (h) vanishes when all spatial momenta $\mathbf{p}_i$ go to zero.

The theory above is an example of "spontaneous" symmetry breaking. The original theory has an $SO(N)$ rotational symmetry among the fields $\mathbf{\Phi}$. After fixing a vacuum state, we only have an $SO(N-1)$ rotational symmetry among the fields $\boldsymbol{\pi}$. The symmetries that rotate Φ_N into the π_i are still there, but operating with them moves us between different vacuum states; they tell us that the vacua all have the same energy. Thus, there is no energy cost for shifting a pion field π_i by a small constant, corresponding to the fact that there is no mass term for the pion fields. This is an example of a Goldstone's theorem, which states that each spontaneously broken continuous symmetry corresponds to a massless boson, called a Goldstone boson.

Pions are the lightest mesons, and mediate interactions between protons and neutrons. The linear sigma model was a phenomenological model which, among other things, explained why the pion was so light. The complete picture we have today is that pions are the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry, a symmetry of quantum chromodynamics which appears for massless up and down quarks.

In reality, the up and down quarks have small masses. This additional, "explicit" breaking of chiral symmetry explains why real pions have nonzero mass. In addition, the quarks differ in electric charge, which explains why the pions have different masses.

j) We can see a similar phenomenon in the linear sigma model. Show that when we add a small term $a\,\Phi_N$ to the Lagrangian, the pion fields get a mass term m_π^2 proportional to a, and find this term. (Hint: to keep things from getting messy, work to lowest order in a as much as possible.)

5. ⋆ A contrived calculation. (5 points)

This problem is optional. In scalar ϕ^3 theory there is a diagram that contributes to the vacuum correlation function at order λ^8, shaped like a cube. Find its symmetry factor. (This is somewhat involved, and the heuristic rules given in lecture will not be enough.)