## 1. Mandelstam variables. (4 points)

Consider any $2 \rightarrow 2$ scattering process, where the two incoming particles have momenta $p_{1}$ and $p_{2}$, and the two outgoing particles have momenta $p_{3}$ and $p_{4}$. By momentum conservation, $p_{1}+p_{2}=p_{3}+p_{4}$, and $p_{i}^{2}=m_{i}^{2}$ where $m_{i}$ is the mass of particle $i$. In this situation, it is often useful to work in terms of the Mandelstam variables

$$
\begin{equation*}
s=\left(p_{1}+p_{2}\right)^{2}, \quad t=\left(p_{1}-p_{3}\right)^{2}, \quad u=\left(p_{1}-p_{4}\right)^{2} . \tag{1}
\end{equation*}
$$

a) Show that $s+t+u=m_{1}^{2}+m_{2}^{2}+m_{3}^{2}+m_{4}^{2}$.

Solution: Just directly using the definitions,

$$
\begin{align*}
s+t+u & =p_{1}^{2}+2 p_{1} \cdot p_{2}+p_{2}^{2}+p_{1}^{2}-2 p_{1} \cdot p_{3}+p_{3}^{2}+p_{1}^{4}-2 p_{1} \cdot p_{4}+p_{4}^{2}  \tag{S1}\\
& =m_{1}^{2}+m_{2}^{2}+m_{3}^{2}+m_{4}^{2}+2 p_{1} \cdot\left(p_{1}+p_{2}-p_{3}-p_{4}\right)  \tag{S2}\\
& =m_{1}^{2}+m_{2}^{2}+m_{3}^{2}+m_{4}^{2} . \tag{S3}
\end{align*}
$$

b) In the center of mass frame, the total energy is $E_{\mathrm{cm}}$ and the angle of $\mathbf{p}_{1}$ to $\mathbf{p}_{3}$ is $\theta$. Write $s, t$, and $u$ in terms of $E_{\mathrm{cm}}$ and $\theta$, assuming all four particles are massless.
Solution: This is easiest if we use concrete coordinate expressions. Aligning the $z$-axis with $\mathbf{p}_{1}$,

$$
\begin{equation*}
p_{1}^{\mu}=\frac{E_{\mathrm{cm}}}{2}(1,0,0,1), \quad p_{2}^{\mu}=\frac{E_{\mathrm{cm}}}{2}(1,0,0,-1) \tag{S4}
\end{equation*}
$$

where the spatial momenta are opposite because we're working in the center of mass frame, and equal in magnitude to the energy because the particles are massless. After the scattering, we have

$$
\begin{equation*}
p_{3}^{\mu}=\frac{E_{\mathrm{cm}}}{2}(1,0, \sin \theta, \cos \theta), \quad p_{4}^{\mu}=\frac{E_{\mathrm{cm}}}{2}(1,0,-\sin \theta,-\cos \theta) . \tag{S5}
\end{equation*}
$$

At this point we can just compute the Mandelstam variables explicitly, giving

$$
\begin{equation*}
s=E_{\mathrm{cm}}^{2}, \quad t=-E_{\mathrm{cm}}^{2} \frac{1-\cos \theta}{2}=-E_{\mathrm{cm}}^{2} \sin ^{2} \frac{\theta}{2}, \quad u=-E_{\mathrm{cm}}^{2} \frac{1+\cos \theta}{2}=-E_{\mathrm{cm}}^{2} \cos ^{2} \frac{\theta}{2} . \tag{S6}
\end{equation*}
$$

Note that they sum to zero, as expected from part (a).

## 2. Scalar Yukawa amplitudes. (7 points)

Let $\phi$ be a real scalar field of mass $M$, and $\psi$ be a complex scalar field of mass $m$, with

$$
\begin{equation*}
\mathcal{L}_{\mathrm{int}}=-g \psi^{*} \psi \phi \tag{2}
\end{equation*}
$$

The field $\psi$ can annihilate a particle, or create a particle with opposite $U(1)$ charge; these particles are conventionally called the $\psi$ and $\psi^{*}$, respectively. Similarly, the field $\phi$ can create or annihilate a $\phi$ particle. (That is, particles are named after the field that annihilates them.) As discussed in section, the Feynman rules are

$$
\begin{equation*}
-----=\frac{i}{p^{2}-M^{2}+i \epsilon} \quad \longrightarrow=\frac{i}{p^{2}-m^{2}+i \epsilon} \quad---\langle=-i g \tag{3}
\end{equation*}
$$

where a dashed line stands for $\phi$ and a solid line stands for $\psi$.
a) Assuming $M>2 m$, calculate the decay rate of a $\phi$ particle to leading order in $g$. (Start from equation (4.86) of Peskin and Schroeder, and do the phase space integrals.)
Solution: The Feynman diagram is just one copy of the interaction vertex, so $i \mathcal{M}=-i g$. Let the initial momentum be $k^{\mu}=(M, \mathbf{0})$ and the final momenta be $p_{1}^{\mu}$ and $p_{2}^{\mu}$. Then

$$
\begin{align*}
\Gamma & =\frac{1}{2 M} \int \frac{d^{3} \mathbf{p}_{1}}{(2 \pi)^{3} 2 E_{1}} \frac{d^{3} \mathbf{p}_{2}}{(2 \pi)^{3} 2 E_{2}}|-i g|^{2}(2 \pi)^{4} \delta^{(4)}\left(k-p_{1}-p_{2}\right)  \tag{S7}\\
& =\frac{g^{2}}{32 \pi^{2} M} \int \frac{d^{3} \mathbf{p}_{1}}{E_{1}} \frac{d^{3} \mathbf{p}_{2}}{E_{2}} \delta^{(4)}\left(k-p_{1}-p_{2}\right)  \tag{S8}\\
& =\frac{g^{2}}{32 \pi^{2} M} \int \frac{d^{3} \mathbf{p}_{1}}{E_{1}^{2}} \delta\left(M-2 E_{1}\right)  \tag{S9}\\
& =\frac{g^{2}}{8 \pi M} \int_{0}^{\infty} d\left|\mathbf{p}_{1}\right| \frac{\left|\mathbf{p}_{1}\right|^{2}}{E_{1}^{2}} \delta\left(M-2 E_{1}\right)  \tag{S10}\\
& =\frac{g^{2}}{8 \pi M} \int_{m}^{\infty} d E_{1} \frac{\left|\mathbf{p}_{1}\right|}{E_{1}} \delta\left(M-2 E_{1}\right)  \tag{S11}\\
& =\frac{g^{2}}{16 \pi M} \sqrt{1-(2 m / M)^{2}} . \tag{S12}
\end{align*}
$$

b) Find the amplitude for $\psi\left(p_{1}\right) \psi^{*}\left(p_{2}\right) \rightarrow \psi\left(p_{3}\right) \psi^{*}\left(p_{4}\right)$ scattering to leading order in $g$, in terms of Mandelstam variables.

Solution: There are two Feynman diagrams, giving amplitude

$$
\begin{align*}
i \mathcal{M} & =  \tag{S13}\\
& =(-i g)^{2}\left(\frac{i}{\left(p_{1}+p_{2}\right)^{2}-M^{2}+i \epsilon}+\frac{i}{\left(p_{1}-p_{3}\right)^{2}-M^{2}+i \epsilon}\right)  \tag{S14}\\
& =-g^{2}\left(\frac{i}{s-M^{2}+i \epsilon}+\frac{i}{t-M^{2}+i \epsilon}\right) . \tag{S15}
\end{align*}
$$

c) Suppose the energies we can reach in an experiment are higher than $m$, but much lower than $M$. In this case, we might not know that $\phi$ particles exist, since we can't produce them, so we would have to describe the scattering process in part (b) using an "effective" field theory in terms of $\psi$ alone. Show that at leading order,

$$
\begin{equation*}
\mathcal{L}_{\text {int }}=-\lambda \psi^{*} \psi \psi^{*} \psi \tag{4}
\end{equation*}
$$

will yield the same answer for part (b) for some value of $\lambda$, and find that value in terms of $g$ and $M$. Assume all elements of the momenta $p_{i}$ are much less than $M$.

Solution: The Feynman rule for the interaction in the effective theory is just


The factor of 4 comes about because there are 2 ways to choose which copy of $\psi$ contracts with what, and similarly 2 ways for $\psi^{*}$.

This is also the amplitude for the scattering process in the effective theory. On the other hand, we have $s, t \ll M$, so the answer to part (b) becomes $2 i g^{2} / M^{2}$. Equating these results gives $\lambda=-g^{2} / 2 M^{2}$. (The fact that $\lambda$ is suppressed by powers of $M$ is a generic feature of effective field theory.)

## 3. Solving a trivial theory. (4 points)

Consider a free real scalar field of mass $m$, but treat the mass term as the perturbation,

$$
\begin{equation*}
\mathcal{L}_{0}=\frac{1}{2}\left(\partial_{\mu} \phi\right)\left(\partial^{\mu} \phi\right), \quad \mathcal{L}_{\text {int }}=-\frac{1}{2} m^{2} \phi^{2} . \tag{5}
\end{equation*}
$$

a) Write down the Feynman rules for this theory.

Solution: The Feynman rules are that the propagator is

$$
\begin{equation*}
\bar{Z}=\frac{i}{p^{2}+i \epsilon} \tag{S17}
\end{equation*}
$$

and the interaction vertex is

$$
\begin{equation*}
\longrightarrow=-i m^{2} \tag{S18}
\end{equation*}
$$

b) Evaluate the momentum-space propagator exactly, summing all of the (infinitely many) connected Feynman diagrams.
Solution: The diagrams are

which correspond to

$$
\begin{equation*}
\frac{i}{p^{2}+i \epsilon}+\frac{i}{p^{2}+i \epsilon}\left(-i m^{2}\right) \frac{i}{p^{2}+i \epsilon}+\frac{i}{p^{2}+i \epsilon}\left(-i m^{2}\right) \frac{i}{p^{2}+i \epsilon}\left(-i m^{2}\right) \frac{i}{p^{2}+i \epsilon}+\ldots . \tag{S20}
\end{equation*}
$$

This is a geometric series where the ratio between terms is $m^{2} /\left(p^{2}+i \epsilon\right)$, so the sum is

$$
\begin{equation*}
\frac{i}{p^{2}+i \epsilon} \frac{1}{1-\frac{m^{2}}{p^{2}+i \epsilon}}=\frac{i}{p^{2}-m^{2}+i \epsilon} . \tag{S21}
\end{equation*}
$$

Of course, this is just the familiar propagator for a free massive scalar field.

## 4. Pion scattering in the linear sigma model. (25 points)

Consider a theory with $N$ free real scalar fields $\Phi_{i}(x)$ of equal mass $m$,

$$
\begin{equation*}
\mathcal{L}_{0}=\sum_{i=1}^{N} \frac{1}{2} \partial_{\mu} \Phi_{i} \partial^{\mu} \Phi_{i}-\frac{m^{2}}{2} \Phi_{i} \Phi_{i} . \tag{6}
\end{equation*}
$$

This Lagrangian is symmetric under rotations of the scalar fields among themselves. We can show this more clearly by defining an $N$-element vector $\Phi$ with elements of $\Phi_{i}$, so

$$
\begin{equation*}
\mathcal{L}_{0}=\frac{1}{2}\left(\partial_{\mu} \boldsymbol{\Phi}\right)^{2}-\frac{m^{2}}{2} \boldsymbol{\Phi} \cdot \boldsymbol{\Phi} . \tag{7}
\end{equation*}
$$

This is just shorthand for Eq. (6). Note that the index on $\Phi_{i}$ is not a spatial index (such as on $\partial_{i}$ ). It is a "flavor" index, meaning it just identifies which field we're talking about.
a) We quantize the theory by imposing the equal time commutation relations

$$
\begin{equation*}
\left[\Pi_{i}(\mathbf{x}, 0), \Phi_{j}(\mathbf{y}, 0)\right]=i \delta_{i j} \delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad \Pi_{i}(x)=\partial_{t} \Phi_{i}(x) \tag{8}
\end{equation*}
$$

with all other commutators vanishing. Show that the propagator is

$$
\begin{equation*}
\langle 0| T\left\{\Phi_{i}(x) \Phi_{j}(y)\right\}|0\rangle=\Phi_{i} \widehat{(x) \Phi_{j}}(y)=\delta_{i j} D_{F}(x-y) \tag{9}
\end{equation*}
$$

where $\delta_{i j}$ is 1 if $i=j$, and 0 otherwise. The momentum space Feynman rule is

$$
\begin{equation*}
i-j=\frac{i \delta_{i j}}{p^{2}-m^{2}+i \epsilon} \tag{10}
\end{equation*}
$$

where the $i$ and $j$ are flavor indices.
Solution: The commutation relations imply that the mode expansion of the field $\Phi_{i}$ involves a set of creation and annihilation operators $a_{i}^{\dagger}(\mathbf{k})$ and $a_{i}(\mathbf{k})$, whose commutators vanish for $i \neq j$. Therefore, when $i \neq j$, the contraction of two fields vanishes because $\Phi_{i}^{+}$and $\Phi_{j}^{-}$automatically commute. When $i=j$, the derivation of the propagator is just the same as for a single real scalar field.
b) The linear sigma model additionally contains the interaction

$$
\begin{equation*}
\mathcal{L}_{\mathrm{int}}=-\frac{\lambda}{4}(\boldsymbol{\Phi} \cdot \boldsymbol{\Phi})^{2} . \tag{11}
\end{equation*}
$$

which is also symmetric under rotations of $\boldsymbol{\Phi}$. Show that this interaction corresponds to the momentum space Feynman rule

(Hint: consider the cases where all four fields have the same flavor, and when two pairs of fields have the same flavor.)
Solution: Expanding out the interaction explicitly gives

$$
\begin{equation*}
\mathcal{L}_{\text {int }}=-\frac{\lambda}{4}\left(\sum_{i} \Phi_{i} \Phi_{i}\right)^{2}=-\frac{\lambda}{4} \sum_{i, j} \Phi_{i} \Phi_{i} \Phi_{j} \Phi_{j}=-\frac{\lambda}{4} \sum_{i} \Phi_{i}^{4}-\frac{\lambda}{2} \sum_{i<j} \Phi_{i}^{2} \Phi_{j}^{2} . \tag{S22}
\end{equation*}
$$

The first term yields a vertex for four fields of the same flavor. Whenever we introduce a copy of this vertex, we get a factor of $4!=24$ because of the ways to choose how to contract the copies of $\Phi_{i}$ with other things, so the overall vertex factor is $-6 i \lambda$.

The second term yields a vertex for four fields where two pairs have the same flavor. These come with an extra factor of $2 \cdot 2=4$ because of the two copies of $\Phi_{i}$ and two copies of $\Phi_{j}$, and thus the overall vertex factor is $-2 i \lambda$.

Now let's check if this is compatible with the provided Feynman rule. When $i=j=k=\ell$, all the Kronecker deltas are equal to 1 , so we get $-6 i \lambda$. When $i=j$ and $k=\ell$, but $i \neq k$, then only the first Kronecker delta is equal to 1 , and the others are equal to 0 , so we get $-2 i \lambda$.
c) Find the total cross section in the centre of mass frame for the processes

$$
\begin{equation*}
\Phi_{1} \Phi_{2} \rightarrow \Phi_{1} \Phi_{2}, \quad \Phi_{1} \Phi_{1} \rightarrow \Phi_{2} \Phi_{2}, \quad \Phi_{1} \Phi_{1} \rightarrow \Phi_{1} \Phi_{1} \tag{13}
\end{equation*}
$$

to leading order in $\lambda$, in terms of $E_{\mathrm{cm}}$. (Hint: start from equation (4.85) of Peskin and Schroeder, and be careful with factors of 2.)
Solution: The Feynman diagrams for all three of these processes are just one copy of the interaction vertex, and the amplitudes are just $-2 i \lambda,-2 i \lambda$, and $-6 i \lambda$ respectively. Because all four masses are identical, we may use equation (4.85), which states

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}=\frac{|\mathcal{M}|^{2}}{64 \pi^{2} E_{\mathrm{cm}}^{2}} . \tag{S23}
\end{equation*}
$$

The scattering is isotropic, so the angular integrals are trivial.
For the first process, the angular integral just gives a factor of $4 \pi$, so

$$
\begin{equation*}
\sigma\left(\Phi_{1} \Phi_{2} \rightarrow \Phi_{1} \Phi_{2}\right)=\frac{4 \pi \lambda^{2}}{16 \pi^{2} E_{\mathrm{cm}}^{2}}=\frac{\lambda^{2}}{4 \pi E_{\mathrm{cm}}^{2}} \tag{S24}
\end{equation*}
$$

For the second process, the angular integral only gives a factor of $2 \pi$ because the particles in the final state are identical. (The state where the first $\Phi_{2}$ exits left and the second exits right is exactly the same as the state where the first $\Phi_{2}$ exits right and the second exits left. So integrating over all possible directions for the momentum of the first $\Phi_{2}$ would be double counting.) Thus,

$$
\begin{equation*}
\sigma\left(\Phi_{1} \Phi_{1} \rightarrow \Phi_{2} \Phi_{2}\right)=\frac{\lambda^{2}}{8 \pi E_{\mathrm{cm}}^{2}} \tag{S25}
\end{equation*}
$$

Finally, the last cross section is 9 times bigger, giving

$$
\begin{equation*}
\sigma\left(\Phi_{1} \Phi_{1} \rightarrow \Phi_{1} \Phi_{1}\right)=\frac{9 \lambda^{2}}{8 \pi E_{\mathrm{cm}}^{2}} \tag{S26}
\end{equation*}
$$

In the linear sigma model, the potential energy of a uniform classical field is

$$
\begin{equation*}
V(\Phi)=\frac{m^{2}}{2} \boldsymbol{\Phi} \cdot \boldsymbol{\Phi}+\frac{\lambda}{4}(\boldsymbol{\Phi} \cdot \boldsymbol{\Phi})^{2} \tag{14}
\end{equation*}
$$

When we quantize the harmonic oscillator, the usual definition of the creation and annihilation operators in terms of $x$ and $p$ only makes sense if the potential's minimum is at $x=0$. If the minimum is somewhere else, then many things go wrong. For instance, $a|0\rangle$ won't be zero, and more generally $a^{\dagger}$ and $a$ won't have simple commutation relations with $H$, so won't properly raise and lower the energy. Similarly, it only makes sense to quantize fields in the usual way about minima of the potential $V(\Phi)$.

If $m^{2}>0$ and $\lambda>0$, there is a unique minimum of the potential at $\boldsymbol{\Phi}=\mathbf{0}$, so our treatment above makes sense. Now suppose that $m^{2}<0$ and $\lambda>0$.
d) Defining $\mu^{2}=-m^{2}$, show that the minima of the potential are at

$$
\begin{equation*}
|\boldsymbol{\Phi}|=v=\sqrt{\frac{\mu^{2}}{\lambda}} \tag{15}
\end{equation*}
$$

The quantum theory thus has multiple vacuum states, but all have nonvanishing $\langle\boldsymbol{\Phi}\rangle$.
Solution: Setting $d V / d|\boldsymbol{\Phi}|$ to zero gives $m^{2}|\boldsymbol{\Phi}|+\lambda|\boldsymbol{\Phi}|^{3}=0$, which yields the desired result.
By symmetry under rotations of $\boldsymbol{\Phi}$, we can suppose without loss of generality that we are in a vacuum state where $\left\langle\Phi_{N}\right\rangle=v$, with all others vanishing. Define the new fields

$$
\begin{equation*}
\sigma(x)=\Phi_{N}(x)-v, \quad \pi_{i}(x)=\Phi_{i}(x), \quad i \in\{1,2, \ldots, N-1\} \tag{16}
\end{equation*}
$$

which all have vanishing expectation values in this vacuum.
e) Show that in terms of these fields, the Lagrangian is

$$
\begin{align*}
\mathcal{L}= & \frac{1}{2}\left(\partial_{\mu} \boldsymbol{\pi}\right)^{2}+\frac{1}{2}\left(\partial_{\mu} \sigma\right)^{2}-\frac{1}{2} m_{\sigma}^{2} \sigma^{2} \\
& \quad-\frac{\lambda_{4 \pi}}{4}(\boldsymbol{\pi} \cdot \boldsymbol{\pi})^{2}-\lambda_{3 \sigma} \sigma^{3}-\frac{\lambda_{4 \sigma}}{4} \sigma^{4}-\lambda_{\pi \sigma 1}(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \sigma-\lambda_{\pi \sigma 2}(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \sigma^{2}+C . \tag{17}
\end{align*}
$$

and give expressions for $m_{\sigma}, \lambda_{4 \pi}, \lambda_{3 \sigma}, \lambda_{4 \sigma}, \lambda_{\pi \sigma 1}, \lambda_{\pi \sigma 2}$ and $C$ in terms of $\mu$ and $\lambda$. We see that $\sigma$ is massive, but we have $N-1$ massless pion fields $\pi_{i}$.
Solution: This is straightforward algebra, and the answers are

$$
\begin{equation*}
m_{\sigma}=\sqrt{2} \mu, \quad \lambda_{4 \pi}=\lambda, \quad \lambda_{3 \sigma}=\mu \sqrt{\lambda}, \quad \lambda_{4 \sigma}=\lambda \tag{S27}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{\pi \sigma 1}=\mu \sqrt{\lambda}, \quad \lambda_{\pi \sigma 2}=\frac{\lambda}{2}, \quad C=-\frac{\mu^{4}}{4 \lambda} . \tag{S28}
\end{equation*}
$$

f) Write the momentum-space Feynman rules in terms of $\mu$ and $\lambda$. Draw the pions with a solid line and flavor index $i$, and the $\sigma$ particle with a double solid line. (Hint: if you've seen the pattern from the other Feynman rules in the problem set, it should be possible to write down the answer with some thought, but no explicit calculation. There is no Feynman rule corresponding to the constant $C$, which has no effect here.)
Solution: The propagators are

$$
\begin{equation*}
i \longrightarrow j=\frac{i \delta_{i j}}{p^{2}+i \epsilon} \quad \overline{=}=\frac{i}{p^{2}-m_{\sigma}^{2}+i \epsilon} \tag{S29}
\end{equation*}
$$

and the interaction vertices are

g) Compute the decay rate of a $\sigma$ particle to leading order in $\lambda$, and give the corresponding lifetime in seconds if $\lambda=0.1, \mu=10 \mathrm{GeV}$ and $N=3$.
Solution: The amplitude to decay to a pair of some species of pion is $i \mathcal{M}=-2 i \mu \sqrt{\lambda}$, so recycling the result to problem 2(a) gives the decay rate

$$
\begin{equation*}
\Gamma=\frac{N-1}{2} \frac{(2 \sqrt{\lambda} \mu)^{2}}{8 \pi m_{\sigma}}=\frac{(N-1) \lambda \mu}{4 \sqrt{2} \pi} . \tag{S32}
\end{equation*}
$$

where the factor of $1 / 2$ accounts for the two identical particles in the final state, and $N-1$ is the number of species of pion. Inverting this and converting back from natural units gives a lifetime $6 \times 10^{-24} \mathrm{~s}$.
h) Find the amplitude for $\pi_{i}\left(p_{1}\right) \pi_{j}\left(p_{2}\right) \rightarrow \pi_{k}\left(p_{3}\right) \pi_{\ell}\left(p_{4}\right)$ scattering to leading order in $\lambda$, in terms of Mandelstam variables. (Hint: there are four Feynman diagrams.)
Solution: The amplitude is

$$
\begin{align*}
i \mathcal{M} & ={ }_{j}^{i}  \tag{S33}\\
& =(-2 i \sqrt{\lambda} \mu)^{2}\left(\frac{i \delta_{i j} \delta_{k \ell}}{s-m_{\sigma}^{2}+i \epsilon}+\frac{i \delta_{i k} \delta_{j \ell}}{t-m_{\sigma}^{2}+i \epsilon}+\frac{i \delta_{i \ell} \delta_{j k}}{u-m_{\sigma}^{2}+i \epsilon}\right)-2 i \lambda\left(\delta_{i j} \delta_{k \ell}+\delta_{i k} \delta_{j \ell}+\delta_{i \ell} \delta_{j k}\right)  \tag{S34}\\
& =(-2 i \lambda)\left(\frac{2 \mu^{2} \delta_{i j} \delta_{k \ell}}{s-2 \mu^{2}+i \epsilon}+\frac{2 \mu^{2} \delta_{i k} \delta_{j \ell}}{t-2 \mu^{2}+i \epsilon}+\frac{2 \mu^{2} \delta_{i \ell} \delta_{j k}}{u-2 \mu^{2}+i \epsilon}+\delta_{i j} \delta_{k \ell}+\delta_{i k} \delta_{j \ell}+\delta_{i \ell} \delta_{j k}\right) . \tag{S35}
\end{align*}
$$

i) Show that the amplitude in part (h) vanishes when all spatial momenta $\mathbf{p}_{i}$ go to zero.

Solution: In this limit, we have $s=t=u=0$, so the denominators all become $-2 \mu^{2}$. Then each of the first three diagrams cancels with one term from the last diagram.

This is another manifestation of Goldstone's theorem, discussed below. The general statement is that in the effective theory of pions alone (i.e. without $\sigma$ ), sometimes called the nonlinear sigma model, the Lagrangian only depends on the spatial derivatives $\partial_{\mu} \pi$ of the pion field. Since acting with a derivative on a field produces a factor of the particle momentum $p^{\mu}$, the Feynman rules for the interactions always come with factors of momenta, so scattering amplitudes like the one we computed must vanish at zero momentum.

The theory above is an example of "spontaneous" symmetry breaking. The original theory has an $S O(N)$ rotational symmetry among the fields $\boldsymbol{\Phi}$. After fixing a vacuum state, we only have an $S O(N-1)$ rotational symmetry among the fields $\pi$. The symmetries that rotate $\Phi_{N}$ into the $\pi_{i}$ are still there, but operating with them moves us between different vacuum states; they tell us that the vacua all have the same energy. Thus, there is no energy cost for shifting a pion field $\pi_{i}$ by a small constant, corresponding to the fact that there is no mass term for the pion fields. This is an example of a Goldstone's theorem, which states that each spontaneously broken continuous symmetry corresponds to a massless boson, called a Goldstone boson.

Pions are the lightest mesons, and mediate interactions between protons and neutrons. The linear sigma model was a phenomenological model which, among other things, explained why the pion was so light. The complete picture we have today is that pions are the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry, a symmetry of quantum chromodynamics which appears for massless up and down quarks.

In reality, the up and down quarks have small masses. This additional, "explicit" breaking of chiral symmetry explains why real pions have nonzero mass. In addition, the quarks differ in electric charge, which explains why the pions have different masses.
j) We can see a similar phenomenon in the linear sigma model. Show that when we add a small term $a \Phi_{N}$ to the Lagrangian, the pion fields get a mass term $m_{\pi}^{2}$ proportional to $a$, and find this term. (Hint: to keep things from getting messy, work to lowest order in $a$ as much as possible.)
Solution: The potential now includes $-a \Phi_{N}$, which tilts it in the $\Phi_{N}$ direction. There is now a unique vacuum state where only $\left\langle\Phi_{N}\right\rangle=v$ is nonzero. Setting the derivative of the potential to zero gives

$$
\begin{equation*}
-\mu^{2} v+\lambda v^{3}-a=0 \tag{S37}
\end{equation*}
$$

which rearranges to

$$
\begin{equation*}
\lambda v^{2}=\mu^{2}+\frac{a}{v} . \tag{S38}
\end{equation*}
$$

To solve this at leading order in $a$, we replace the $v$ on the right-hand side with the unperturbed expectation value $v_{0}=\sqrt{\mu^{2} / \lambda}$. Solving for $v$ then yields

$$
\begin{equation*}
v=\sqrt{\frac{\mu^{2}}{\lambda}+\frac{a}{\mu \sqrt{\lambda}}}=v_{0}+\frac{a}{2 \mu^{2}}+O\left(a^{2}\right) . \tag{S39}
\end{equation*}
$$

To find the mass term for the pion, it's easiest to start with Eq. 17) and do a second shift, defining

$$
\begin{equation*}
\sigma^{\prime}=\sigma-\frac{a}{2 \mu^{2}} \tag{S40}
\end{equation*}
$$

The only term that will produce a pion mass term linear in $a$ is $-\lambda_{\pi \sigma 1}(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \sigma$, giving

$$
\begin{equation*}
\mathcal{L} \supset-\mu \sqrt{\lambda}(\boldsymbol{\pi} \cdot \boldsymbol{\pi}) \frac{a}{2 \mu^{2}}=-\frac{a \sqrt{\lambda}}{2 \mu} \boldsymbol{\pi} \cdot \boldsymbol{\pi} . \tag{S41}
\end{equation*}
$$

In other words, the pion mass is $\sqrt{a \sqrt{\lambda} / \mu}$.

## 5. $\star$ A contrived calculation. (5 points)

This problem is optional. In scalar $\phi^{3}$ theory there is a diagram that contributes to the vacuum correlation function at order $\lambda^{8}$, shaped like a cube. Find its symmetry factor. (This is somewhat involved, and the heuristic rules given in lecture will not be enough.)

Solution: The most reliable way to do this is to simply count all the Wick contractions that yield a cube, building up the diagram one vertex at a time.

- Consider what the three copies of $\phi$ in the first vertex can contract with. The first has $7 \cdot 3$ choices, the second has $6 \cdot 3$ choices, and the third has $5 \cdot 3$ choices.
- Now consider the opposite corner of the cube. There are 4 choices for which remaining uncontracted vertex is the opposite corner. Its three copies of $\phi$ have $3 \cdot 3,2 \cdot 3$, and 3 choices for contractions.
- Finally, we attach the two halves of the cube together. There are 6 ways of choosing which vertices attach to which, and $2^{6}$ choices for how $\phi$ fields are contracted to make the joining edges.

Therefore, the symmetry factor is

$$
\begin{equation*}
\frac{7 \cdot 3 \cdot 6 \cdot 3 \cdot 5 \cdot 3 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 3 \cdot 3 \cdot 6 \cdot 2^{6}}{8!\cdot(3!)^{8}}=\frac{1}{48} \tag{S42}
\end{equation*}
$$

This makes sense, as the symmetry group of the cube has 48 elements.

