1. The Lorentz algebra. (10 points)

You previously showed the rotation and boost generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{1}$$

These are Euclidean indices, which is why pairs of upstairs indices can be contracted.

a) Defining $J^i_\pm = (J^i \pm iK^i)/2$, show that

$$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm, \tag{2}$$

so that the Lorentz algebra is just two copies of the algebra of the rotation group.

The commutation relations can be expressed more concisely if we collect all six generators into the antisymmetric Lorentz tensor $J^{\mu\nu}$, where

$$J^{ij} = \epsilon^{ijk} J^k, \qquad J^{0i} = K^i \tag{3}$$

The commutation relations (1) can then all be written together as

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}). \tag{4}$$

By definition, a representation of the Lorentz algebra is a choice of $J^{\mu\nu}$ that satisfies these commutation relations. The simplest example is the trivial representation $J^{\mu\nu} = 0$, which describes the action of Lorentz transformations on scalars.

b) In the vector representation, each generator $J^{\mu\nu}$ is a 4×4 matrix, with elements

$$(J^{\mu\nu})_{\alpha\beta} = i\,(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta). \tag{5}$$

This describes the action of Lorentz transformations on four-vectors V^β, as you've already seen. Starting from (5), verify the generators satisfy the Lorentz algebra (4). (Hint: upper and lower indices should be contracted in pairs to perform matrix multiplication. For example, $(J^{01} J^{23})_{\alpha\beta} = (J^{01})_\alpha{}^\gamma (J^{23})_{\gamma\beta}$. If you get confused, start by plugging in specific values for the indices and doing the sums explicitly.)

c) The gamma matrices are 4×4 matrices defined to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}. \tag{6}$$

Show that

$$[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] = 2\,(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu). \tag{7}$$

d) In the Dirac spinor representation, the generators are the 4×4 matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{8}$$

This describes the action of Lorentz transformations on Dirac spinors. Show that these generators satisfy the Lorentz algebra (4).

2. Properties of gamma matrices. (15 points)

In this problem, you should only use the defining property (6) of the gamma matrices. Below, $\mathbb{1}_4$ denotes a 4×4 identity matrix.

a) Show that contractions of gamma matrices satisfy

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 4\,\mathbb{1}_4 && (9)\\
\gamma^\mu \gamma^\nu \gamma_\mu &= -2\,\gamma^\nu && (10)\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= 4\,\eta^{\nu\rho}\,\mathbb{1}_4 && (11)\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= -2\,\gamma^\sigma \gamma^\rho \gamma^\nu && (12)
\end{aligned}$$

b) Show that the traces of products of gamma matrices obey

$$\begin{aligned}
\operatorname{tr} \gamma^\mu &= 0 && (13)\\
\operatorname{tr} \gamma^\mu \gamma^\nu &= 4\,\eta^{\mu\nu} && (14)\\
\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho &= 0 && (15)\\
\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). && (16)
\end{aligned}$$

c) It will be useful to introduce the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\begin{aligned}
\{\gamma^5, \gamma^\mu\} &= 0 && (17)\\
(\gamma^5)^2 &= \mathbb{1}_4 && (18)\\
\operatorname{tr} \gamma^5 &= 0 && (19)\\
\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^5 &= 0 && (20)\\
\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 &= -4i\,\epsilon^{\mu\nu\rho\sigma}. && (21)
\end{aligned}$$

3. Invariance of the Dirac Lagrangian. (15 points)

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m\mathbb{1}_4)\Psi \qquad (22)$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. The four-component spinor Ψ is acted on by the gamma matrices. In general, a Lorentz transformation Λ will change a spinor according to

$$\Psi(x) \to \Psi'(x') = U(\Lambda)\Psi(x) \qquad (23)$$

where $U(\Lambda)$ is some 4×4 matrix, not necessarily unitary.

a) Show that the Dirac Lagrangian is invariant under Lorentz transformations if

$$U^{-1}(\Lambda) = \gamma^0 U^\dagger(\Lambda)\gamma^0, \qquad U^{-1}(\Lambda)\gamma^\mu U(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \qquad (24)$$

b) Show that for an infinitesimal Lorentz transformation $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ the above relations are satisfied for

$$U(\Lambda) = \mathbb{1}_4 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}, \qquad (25)$$

where $S^{\mu\nu}$ is the Lorentz generator defined by (8). (Hint: use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which holds in every representation of the gamma matrices.)

c) A finite Lorentz transformation is given by exponentiating a generator,

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega^{\mu\nu} S_{\mu\nu}\right). \tag{26}$$

Explicitly write down the 4×4 matrix $U(\Lambda)$ for a rotation about the x-axis by an angle θ, and a boost along the z axis with rapidity ϕ. Use the Dirac representation of the gamma matrices, as this will yield simple results in the nonrelativistic limit.

d) Show that the Dirac Lagrangian is invariant under $\Psi \to e^{-i\alpha}\Psi$, and find the associated conserved current J_V^μ. Then show explicitly that $\partial_\mu J_V^\mu = 0$ using the Dirac equation.

e) Show that when $m = 0$, the Dirac Lagrangian is also invariant under $\Psi \to e^{-i\alpha\gamma^5}\Psi$, and find the associated conserved current J_A^μ. What is $\partial_\mu J_A^\mu$ when m is nonzero?

4. ⋆ Spinors in three dimensions. (5 points)

In this course, we focus on spinors in four dimensions for good reason. In this optional problem, you'll see how the same mathematical structures appear in three dimensions.

a) Consider spinors in three spacetime dimensions. What are the smallest nonzero matrices that can satisfy (6)? Write down three such matrices γ^0, γ^1, and γ^2 explicitly.

b) We define the spinor Lorentz generators by (8) in any dimension. Since there are now only two spatial dimensions, there is only one rotation generator S^{12}. What phase does a spinor pick up after a 2π rotation?

c) How does the tensor product of two spinor representations decompose into irreducible representations of the Lorentz group?

In two spatial dimensions there are exotic particles called anyons, which can pick up an arbitrary phase after a 2π rotation. However, they can't be described by the conventional quantum fields covered in this course.