PHYS-330: QFT I — Solutions 6

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1. The Lorentz algebra. (10 points)

You previously showed the rotation and boost generators obey commutation relations

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \qquad [J^i, K^j] = i\epsilon^{ijk} K^k, \qquad [K^i, K^j] = -i\epsilon^{ijk} J^k. \tag{1}$$

These are Euclidean indices, which is why pairs of upstairs indices can be contracted.

a) Defining $J^i_\pm = (J^i \pm iK^i)/2$, show that

$$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm, \tag{2}$$

so that the Lorentz algebra is just two copies of the algebra of the rotation group.

Solution: In general, for any numbers a and b,

$$\frac{1}{4}\left[J^i + iaK^i, J^j + ibK^j\right] = \frac{1}{4}\left([J^i, J^j] + ia[K^i, J^j] + ib[J^i, K^j] - ab[K^i, K^j]\right) \tag{S1}$$

$$= \frac{i}{4}\epsilon^{ijk}\left((1+ab)J^k + i(a+b)K^k\right). \tag{S2}$$

For $a = -b = 1$ we find

$$[J^i_+, J^j_-] = 0 \tag{S3}$$

as desired. For $a = b = \pm 1$ we find

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm \tag{S4}$$

also as desired.

The commutation relations can be expressed more concisely if we collect all six generators into the antisymmetric Lorentz tensor $J^{\mu\nu}$, where

$$J^{ij} = \epsilon^{ijk} J^k, \qquad J^{0i} = K^i \tag{3}$$

The commutation relations (1) can then all be written together as

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}). \tag{4}$$

By definition, a representation of the Lorentz algebra is a choice of $J^{\mu\nu}$ that satisfies these commutation relations. The simplest example is the trivial representation $J^{\mu\nu} = 0$, which describes the action of Lorentz transformations on scalars.

b) In the vector representation, each generator $J^{\mu\nu}$ is a 4×4 matrix, with elements

$$(J^{\mu\nu})_{\alpha\beta} = i\,(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta). \tag{5}$$

This describes the action of Lorentz transformations on four-vectors V^β, as you've already seen. Starting from (5), verify the generators satisfy the Lorentz algebra (4). (Hint: upper and lower indices should be contracted in pairs to perform matrix multiplication. For example, $(J^{01} J^{23})_{\alpha\beta} = (J^{01})_\alpha{}^\gamma (J^{23})_{\gamma\beta}$. If you get confused, start by plugging in specific values for the indices and doing the sums explicitly.)

Solution: We just directly evaluate the commutator,

$$[J^{\mu\nu}, J^{\rho\sigma}]_{\alpha\delta} = \eta^{\beta\gamma}\left[-(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta)(\delta^\rho_\gamma \delta^\sigma_\delta - \delta^\sigma_\gamma \delta^\rho_\delta) + \delta^\rho_\alpha \delta^\sigma_\beta - \delta^\sigma_\alpha \delta^\rho_\beta)(\delta^\mu_\gamma \delta^\nu_\delta - \delta^\nu_\gamma \delta^\mu_\delta)\right]. \tag{S5}$$

There are 8 terms in total, each of which has the form $\eta^{\beta\gamma}$ times a product of four delta functions. For each one, we can apply the identity $\delta^\mu_\alpha \delta^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$ to reduce the term to a product of two delta functions times a metric tensor. Collecting the terms in four pairs yields the desired result.

This is tedious to write out. An easier way is to note that (S5) is antisymmetric in μ and ν, and in ρ and σ, which means the result also has to be antisymmetric in these pairs of indices. In this way, you can reduce the amount of algebra needed by a factor of 4, though you have to be a bit careful with signs.

c) The gamma matrices are 4×4 matrices defined to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}. \tag{6}$$

Show that

$$[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] = 2\,(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu). \tag{7}$$

Solution: We just directly use the definition of the commutator, and then use (6) repeatedly to get the two terms in the same form, picking up factors of the metric with each anticommutation:

$$\begin{aligned}
[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] &= \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu & \text{(S6)}\\
&= \gamma^\mu\{\gamma^\nu, \gamma^\rho\}\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu & \text{(S7)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - \gamma^\mu\gamma^\rho\{\gamma^\nu, \gamma^\sigma\} + \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu & \text{(S8)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + \{\gamma^\mu, \gamma^\rho\}\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu & \text{(S9)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + 2\eta^{\mu\rho}\gamma^\sigma\gamma^\nu - \gamma^\rho\{\gamma^\mu, \gamma^\sigma\}\gamma^\nu & \text{(S10)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\nu\sigma}\gamma^\mu\gamma^\rho + 2\eta^{\mu\rho}\gamma^\sigma\gamma^\nu - 2\eta^{\mu\sigma}\gamma^\rho\ \gamma^\nu & \text{(S11)}\\
&= 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + 2\eta^{\nu\sigma}\gamma^\rho\gamma^\mu - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma - 2\eta^{\mu\sigma}\gamma^\rho\ \gamma^\nu. & \text{(S12)}
\end{aligned}$$

Note that in the last step, we used the anticommutation relation on each of the middle two terms; the factors of $\eta^{\nu\sigma}\eta^{\mu\rho}$ cancel between the two terms.

d) In the Dirac spinor representation, the generators are the 4×4 matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{8}$$

This describes the action of Lorentz transformations on Dirac spinors. Show that these generators satisfy the Lorentz algebra (4).

Solution: Plugging in the definitions, we have

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] & \text{(S13)}\\
&= -\frac{1}{16}\left([\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] - [\gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu\gamma^\nu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho]\right) & \text{(S14)}\\
&= -\frac{1}{4}[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] & \text{(S15)}\\
&= -\frac{1}{2}\left(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu\right). & \text{(S16)}
\end{aligned}$$

In the third step, we used the anticommutation relations on the last three terms, and in the fourth step we used the result of part (c). To finish the problem, we note that

$$\gamma^\mu\gamma^\nu = \frac{1}{2}[\gamma^\mu, \gamma^\nu] + \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = -2iS^{\mu\nu} + \eta^{\mu\nu}. \tag{S17}$$

Applying this to all four terms, the factors of the metric all cancel out, leaving the desired result

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\eta^{\nu\rho}S^{\mu\sigma} + i\eta^{\nu\sigma}S^{\rho\mu} - i\eta^{\mu\rho}S^\nu\sigma - i\eta^{\mu\sigma}S^{\rho\nu}. \tag{S18}$$

2. Properties of gamma matrices. (15 points)

In this problem, you should only use the defining property (6) of the gamma matrices. Below, $\mathbb{1}_4$ denotes a 4×4 identity matrix.

a) Show that contractions of gamma matrices satisfy

$$\gamma^\mu \gamma_\mu = 4\,\mathbb{1}_4 \tag{9}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\,\gamma^\nu \tag{10}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\,\eta^{\nu\rho}\,\mathbb{1}_4 \tag{11}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\,\gamma^\sigma \gamma^\rho \gamma^\nu \tag{12}$$

Solution: Since γ^μ and γ_μ commute, we have

$$\gamma^\mu \gamma_\mu = \frac{1}{2}(\gamma^\mu \gamma_\mu + \gamma_\mu \gamma^\mu) = \frac{1}{2}\{\gamma^\mu, \gamma_\mu\} = \eta^\mu_\mu = 4 \tag{S19}$$

where the right-hand side has an implicit 4×4 spinor identity matrix. Now, the proofs of the other three statements are very similar, but we use the defining anticommutation relation (6) to get the γ^μ and γ_μ next to each other, so we can use (9). The results are:

$$\begin{aligned}
\gamma^\mu \gamma^\nu \gamma_\mu &= \gamma^\mu \{\gamma^\nu, \gamma_\mu\} - \gamma^\mu \gamma_\mu \gamma^\nu && \text{(S20)}\\
&= 2\gamma^\mu \eta^\nu_\mu - 4\gamma^\nu && \text{(S21)}\\
&= -2\gamma^\nu. && \text{(S22)}\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu &= \gamma^\mu \gamma^\nu \{\gamma^\rho, \gamma_\mu\} - \gamma^\mu \gamma^\nu \gamma_\mu \gamma^\rho && \text{(S23)}\\
&= 2\eta^\rho_\mu \gamma^\mu \gamma^\nu - \gamma^\mu \{\gamma^\nu, \gamma_\mu\} \gamma^\rho + \gamma^\mu \gamma_\mu, \gamma^\nu \gamma^\rho && \text{(S24)}\\
&= 2\gamma^\rho \gamma^\nu - 2\gamma^\nu \gamma^\rho + 4\gamma^\nu \gamma^\rho && \text{(S25)}\\
&= 4\eta^{\rho\nu}. && \text{(S26)}\\
\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu &= 2\eta^\sigma_\mu \gamma^\mu \gamma^\nu \gamma^\rho - \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu \gamma^\sigma && \text{(S27)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\eta^\rho_\mu \gamma^\mu \gamma^\nu \gamma^\sigma + \gamma^\mu \gamma^\nu \gamma_\mu \gamma^\rho \gamma^\sigma && \text{(S28)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\gamma^\rho \gamma^\nu \gamma^\sigma + 2\eta^\nu_\mu \gamma^\mu \gamma^\rho \gamma^\sigma - \gamma^\mu \gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma && \text{(S29)}\\
&= 2\gamma^\sigma \gamma^\nu \gamma^\rho - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\gamma^\nu \gamma^\rho \gamma^\sigma && \text{(S30)}\\
&= 2\gamma^\sigma \{\gamma^\nu, \gamma^\rho\} - 2\gamma^\sigma \gamma^\rho \gamma^\nu - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\{\gamma^\nu, \gamma^\rho\} \gamma^\sigma && \text{(S31)}\\
&= 2\gamma^\sigma \{\gamma^\nu, \gamma^\rho\} - 2\gamma^\sigma \gamma^\rho \gamma^\nu - 2\gamma^\rho \gamma^\nu \gamma^\sigma - 2\{\gamma^\nu, \gamma^\rho\} \gamma^\sigma + 2\gamma^\rho \gamma^\nu \gamma^\sigma && \text{(S32)}\\
&= -2\gamma^\sigma \gamma^\rho \gamma^\nu && \text{(S33)}
\end{aligned}$$

b) Show that the traces of products of gamma matrices obey

$$\operatorname{tr} \gamma^\mu = 0 \tag{13}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu = 4\,\eta^{\mu\nu} \tag{14}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho = 0 \tag{15}$$
$$\operatorname{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4\,(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). \tag{16}$$

Solution: We use the cyclic property of the trace,

$$\operatorname{tr} ABC \ldots XYZ = \operatorname{tr} BC \ldots XYZA \tag{S34}$$

as well as the results we proved in part (a). For example, we have

$$\operatorname{tr} \gamma^\mu = \operatorname{tr}\left[\frac{1}{4}\gamma^\mu \gamma^\nu \gamma_\nu\right] = \operatorname{tr}\left[\frac{1}{4}\gamma_\nu \gamma^\mu \gamma^\nu\right] = -\frac{1}{2}\operatorname{tr} \gamma^\mu \tag{S35}$$

where we used (9), the cyclic property of the trace, and (10). The only way this equation can be satisfied is if $\operatorname{tr}\gamma^\mu = 0$, as desired. A similar idea works for the trace of three gamma matrices,

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{4}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\sigma = \frac{1}{4}\operatorname{tr}\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -\frac{1}{2}\operatorname{tr}\gamma^\rho\gamma^\nu\gamma^\mu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu. \tag{S36}$$

We can bring the right-hand side to the same form as the left-hand side using the anticommutation relation,

$$-\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\{\gamma^\rho,\gamma^\nu\} + \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho \tag{S37}$$

where the second step uses (13). We thus conclude $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0$. One can generalize this argument to show that the trace of an odd number of gamma matrices always vanishes. (Of course, there are multiple ways to show this; you could also have done it by inserting factors of $(\gamma^5)^2 = 1$, as in Peskin.)

Now let's consider the equations with an even number of gamma matrices. For two gamma matrices, we can use the cyclic property of the trace to produce an anticommutator, giving

$$\operatorname{tr}\gamma^\mu\gamma^\nu = \frac{1}{2}\operatorname{tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] = \operatorname{tr}[\eta^{\mu\nu}] = 4\eta^{\mu\nu}. \tag{S38}$$

For four gamma matrices, we can use a similar idea, repeatedly using the anticommutation relations to bring γ^μ to the right and then moving it back to the left using the cyclic property of the trace,

$$\begin{aligned}
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= \operatorname{tr}\{\gamma^\mu,\gamma^\nu\}\gamma^\rho\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma && \text{(S39)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - \operatorname{tr}\gamma^\nu\{\gamma^\mu,\gamma^\rho\}\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma && \text{(S40)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + \operatorname{tr}\gamma^\nu\gamma^\rho\{\gamma^\mu,\gamma^\sigma\} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma && \text{(S41)}\\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\rho\nu} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. && \text{(S42)}
\end{aligned}$$

Adding $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ to both sides and dividing by two gives the result.

While these relations might seem random, there's a very simple way to summarize what they mean. Consider an arbitrarily long product of arbitrary gamma matrices,

$$\operatorname{tr}(\gamma^1\gamma^3\gamma^0\gamma^0\gamma^2\gamma^3\gamma^2\gamma^1\cdots). \tag{S43}$$

The defining anticommutation relations tells us that we pick up a minus sign when we swap the order of any two adjacent, distinct gamma matrices. So such a trace can always be rewritten in the form

$$\pm\operatorname{tr}((\gamma^0)^{n_0}(\gamma^1)^{n_1}(\gamma^2)^{n_2}(\gamma^3)^{n_3}) \tag{S44}$$

where n_0 through n_3 are nonnegative integers. The anticommutation relations also tell us that $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$, which reduces the trace above (up to a sign) to

$$\pm\operatorname{tr}((\gamma^0)^{m_0}(\gamma^1)^{m_1}(\gamma^2)^{m_2}(\gamma^3)^{m_3}) \tag{S45}$$

where the m_i are 0 or 1, depending on the parity of the n_i. Looking back at what we've just proven, (13) says the trace of any single gamma matrix is zero, (14) says the trace of the product of any two distinct gamma matrices is zero, and (15) and (16) say the analogous results for three and four gamma matrices. Therefore, the trace (S45) is automatically zero unless *all* of the m_i are equal to zero. The punchline is that the only way to get a nonzero trace is to multiply an even number of copies of each type of gamma matrix.

c) It will be useful to introduce the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\begin{aligned}
\{\gamma^5,\gamma^\mu\} &= 0 && (17)\\
(\gamma^5)^2 &= \mathbb{1}_4 && (18)\\
\operatorname{tr}\gamma^5 &= 0 && (19)\\
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^5 &= 0 && (20)\\
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 &= -4i\,\epsilon^{\mu\nu\rho\sigma}. && (21)
\end{aligned}$$

Solution: We can check the first statement for each value of μ. For example, for $\mu = 0$ we have

$$\{\gamma^5, \gamma^0\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \tag{S46}$$

The second term can be brought to the first term by performing three anticommutations, each of which flips the sign. Thus, the sum of the two terms is zero. The logic for $\mu = 1, 2, 3$ is similar.

To prove the second statement, we note that

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 &\text{(S47)}\\ &= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 &\text{(S48)}\\ &= -\gamma^2\gamma^3\gamma^2\gamma^3 &\text{(S49)}\\ &= -\gamma^3\gamma^3 &\text{(S50)}\\ &= 1 &\text{(S51)}\end{aligned}$$

where we just used the anticommutation relations, which imply $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$.

To prove the third statement, we can simply use (16),

$$\operatorname{tr}\gamma^5 = i\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3 = 4i(\eta^{01}\eta^{23} - \eta^{02}\eta^{13} + \eta^{03}\eta^{12}) = 0. \tag{S52}$$

For the fourth statement, we can do some casework on the values of μ and ν. If $\mu = \nu$, then the γ^μ and γ^ν multiply to give ± 1, leaving us with $\operatorname{tr}\gamma^5 = 0$. On the other hand, suppose $\mu \neq \nu$, then we can bring the factors of γ^μ and γ^ν within γ^5 to the left, by performing anticommutations. Multiplying them with $\gamma^\mu\gamma^\nu$ yields ± 1 and leaves the trace of a product of two different gamma matrices, which is zero.

Finally, for the last statement, let's first show that the left-hand side is totally antisymmetric in its four indices. For the μ and ν indices, note that

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = 2\eta^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma\gamma^5 - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5 = -\operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5. \tag{S53}$$

By similar reasoning, we get a sign flip when we exchange the ν and ρ indices, and the ρ and σ indices, which suffices to show that the left-hand side is antisymmetric in all four indices. Thus, it must be equal to $A\epsilon^{\mu\nu\rho\sigma}$ for some constant A. To find the constant, we just consider a special case,

$$\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = A\epsilon^{0123} = A. \tag{S54}$$

The left-hand side is just

$$-i\operatorname{tr}(\gamma^5)^2 = -4i \tag{S55}$$

which yields the desired result.

3. Invariance of the Dirac Lagrangian. (15 points)

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m\mathbb{1}_4)\Psi \tag{22}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. The four-component spinor Ψ is acted on by the gamma matrices. In general, a Lorentz transformation Λ will change a spinor according to

$$\Psi(x) \to \Psi'(x') = U(\Lambda)\Psi(x) \tag{23}$$

where $U(\Lambda)$ is some 4×4 matrix, not necessarily unitary.

a) Show that the Dirac Lagrangian is invariant under Lorentz transformations if

$$U^{-1}(\Lambda) = \gamma^0 U^\dagger(\Lambda)\gamma^0, \qquad U^{-1}(\Lambda)\gamma^\mu U(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{24}$$

Solution: From basic special relativity, we know that under a Lorentz transformation,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad \frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x^\nu}. \tag{S56}$$

Now let's show that each term in the Dirac Lagrangian is invariant. The mass term is invariant because

$$\bar{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi \to \Psi^\dagger U^\dagger \gamma^0 U\Psi = \Psi^\dagger\gamma^0 U^{-1}U\Psi = \bar{\Psi}\Psi. \tag{S57}$$

To show the kinetic term is invariant, note that

$$\begin{aligned}
\bar{\Psi}\partial\!\!\!/\Psi &\to \Psi^\dagger U^\dagger \gamma^0 \gamma_\mu \Lambda^\mu{}_\nu \partial^\nu U\Psi && \text{(S58)}\\
&= \Psi^\dagger\gamma^0 U^{-1}\gamma_\mu \Lambda^\mu{}_\nu U \partial^\nu \Psi && \text{(S59)}\\
&= \bar{\Psi}\Lambda_{\mu\rho}\gamma^\rho \Lambda^\mu{}_\nu \partial^\nu\Psi && \text{(S60)}\\
&= \bar{\Psi}\eta_{\nu\rho}\gamma^\rho\partial^\nu\Psi && \text{(S61)}\\
&= \bar{\Psi}\partial\!\!\!/\Psi. && \text{(S62)}
\end{aligned}$$

b) Show that for an infinitesimal Lorentz transformation $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ the above relations are satisfied for

$$U(\Lambda) = \mathbb{1}_4 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}, \tag{25}$$

where $S^{\mu\nu}$ is the Lorentz generator defined by (8). (Hint: use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which holds in every representation of the gamma matrices.)

Solution: Expanding out the definitions, we have

$$U = \mathbb{1} + \frac{1}{8}[\gamma^\mu, \gamma^\nu]\omega_{\mu\nu} \tag{S63}$$

which implies

$$U^{-1} = \mathbb{1} - \frac{1}{8}[\gamma^\mu, \gamma^\nu]\omega_{\mu\nu} \tag{S64}$$

since we're considering infinitesimal translations. To prove the first result note that

$$\begin{aligned}
\gamma^0 U^\dagger \gamma^0 &= \mathbb{1} + \frac{1}{8}\gamma^0[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger]\gamma^0\,\omega_{\mu\nu} && \text{(S65)}\\
&= \mathbb{1} + \frac{1}{8}\gamma^0[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0]\gamma^0\,\omega_{\mu\nu} && \text{(S66)}\\
&= \mathbb{1} - \frac{1}{8}[\gamma^\mu, \gamma^\nu]\omega_{\mu\nu} && \text{(S67)}\\
&= U^{-1}. && \text{(S68)}
\end{aligned}$$

In the first step, we used the fact that $[A, B]^\dagger = [B^\dagger, A^\dagger]$, in the second step we used $\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0$, and in the third step we used $\gamma^0\gamma^0 = 1$.

For the second relation, note that

$$\gamma_\mu[\gamma_\rho, \gamma_\sigma] = 4(\eta_{\mu\rho}\gamma_\sigma - \eta_{\mu\sigma}\gamma_\rho) + [\gamma_\rho, \gamma_\sigma]\gamma_\mu. \tag{S69}$$

Next,

$$\gamma^\mu U = \gamma^\mu(\mathbb{1} + \frac{1}{8}[\gamma_\rho, \gamma_\sigma]\omega^{\rho\sigma}) = U\gamma^\mu + \frac{1}{2}(\omega^{\mu\sigma}\gamma_\sigma - \omega^{\sigma\mu}\gamma_\sigma) = U\gamma^\mu + \omega^{\mu\sigma}\gamma_\sigma. \tag{S70}$$

Finally, we find

$$U^{-1}\gamma_\mu U = U^{-1}(U\gamma_\mu + \omega_{\mu\sigma}\gamma^\sigma) = (\eta_{\mu\nu} + \omega_{\mu\nu})\gamma^\nu = \Lambda_{\mu\nu}\gamma^\nu \tag{S71}$$

as desired. Note that in the penultimate step, we dropped a term of order ω^2 since the Lorentz transformation is infinitesimal.

c) A finite Lorentz transformation is given by exponentiating a generator,

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\right). \tag{26}$$

Explicitly write down the 4×4 matrix $U(\Lambda)$ for a rotation about the x-axis by an angle θ, and a boost along the z axis with rapidity ϕ. Use the Dirac representation of the gamma matrices, as this will yield simple results in the nonrelativistic limit.

Solution: For rotations around the x-axis by an angle θ we have $\omega^{23} = -\omega^{32} = \theta$ and all other entries zero. In the Dirac representation, we find

$$[\gamma^2, \gamma^3] = \left[\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\right] = -\begin{pmatrix} [\sigma_2, \sigma_3] & 0 \\ 0 & [\sigma_2, \sigma_3] \end{pmatrix} = 2i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \tag{S72}$$

Consequently,

$$U(\Lambda(\theta)) = \exp\left\{\frac{i\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\right\} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{S73}$$

This makes sense, because it just says that both of the two-component spinors that make up a Dirac spinor transform like spin $1/2$ particles in ordinary quantum mechanics.

For boosts along the z direction we have $\omega^{03} = -\omega^{30} = \phi$ and all other entries zero, and

$$[\gamma^0, \gamma^3] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\right] = 2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{S74}$$

Consequently,

$$U(\Lambda(\phi)) = \exp\left\{\frac{\phi}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right\} = \cosh\frac{\phi}{2}\mathbb{1} + \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\sinh\frac{\phi}{2}. \tag{S75}$$

In other words, boosts mix the two two-component spinors, while keeping the spin state the same.

d) Show that the Dirac Lagrangian is invariant under $\Psi \to e^{-i\alpha}\Psi$, and find the associated conserved current J_V^μ. Then show explicitly that $\partial_\mu J_V^\mu = 0$ using the Dirac equation.

Solution: The Lagrangian is invariant because we also have $\bar{\Psi} \to e^{i\alpha}\bar{\Psi}$, so the phases cancel out. Now, to avoid confusion when applying Noether's theorem, let's explicitly write out the spinor indices. The changes in the spinors are

$$(\delta\Psi)_a = -i\Psi_a, \qquad (\delta\bar{\Psi})_a = (i\bar{\Psi})_a \tag{S76}$$

and Noether's theorem states

$$J_V^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)_a}(\delta\Psi)_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})_a}(\delta\bar{\Psi})_a \tag{S77}$$

where there is an implicit sum over the spinor index a. (It ranges from 0 to 3, covering the four elements of the spinor, but it is not a Lorentz index.) The second term is just zero, so we get

$$J_V^\mu = (i\bar{\Psi}\gamma^\mu)_a(-i\Psi)_a. \tag{S78}$$

We can rewrite this without explicit spinor indices as

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{S79}$$

where there are now implicit spinor matrix multiplications. To check this is conserved, note that

$$\partial_\mu J_V^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = im\bar{\Psi}\Psi - im\bar{\Psi}\Psi = 0. \tag{S80}$$

e) Show that when $m = 0$, the Dirac Lagrangian is also invariant under $\Psi \to e^{-i\alpha\gamma^5}\Psi$, and find the associated conserved current J_A^μ. What is $\partial_\mu J_A^\mu$ when m is nonzero?

Solution: Using the fact that γ^5 anticommutes with all gamma matrices, we have

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \to \Psi^\dagger e^{i\alpha\gamma^5} \gamma^0 = \Psi^\dagger \gamma^0 e^{-i\alpha\gamma^5} = \bar{\Psi} e^{-i\alpha\gamma^5}. \tag{S81}$$

Thus, the Lagrangian becomes

$$\Psi^\dagger e^{-i\alpha\gamma^5} \not{\partial} e^{-i\alpha\gamma^5} \Psi. \tag{S82}$$

Moving the $e^{-i\alpha\gamma^5}$ past the $\not{\partial}$ flips the sign of the exponential, so we end up with a factor of $e^{i\alpha\gamma^5} e^{-i\alpha\gamma^5} = 1$. Now, the changes in the spinors per α, for infinitesimal α, are

$$\delta\Psi = -i\gamma^5\Psi, \qquad \delta\bar{\Psi} = -i\bar{\Psi}\gamma^5 \tag{S83}$$

which means the Noether current is

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi. \tag{S84}$$

For nonzero m, the divergence of the current is

$$\begin{aligned}
\partial_\mu J_A^\mu &= (\partial_\mu \bar{\Psi})\gamma^\mu\gamma^5\Psi + \bar{\Psi}\gamma^\mu\gamma^5\partial_\mu\Psi && \text{(S85)}\\
&= (\partial_\mu \bar{\Psi})\gamma^\mu\gamma^5\Psi - \bar{\Psi}\gamma^5\gamma^\mu\partial_\mu\Psi && \text{(S86)}\\
&= im\bar{\Psi}\gamma^5\Psi + im\bar{\Psi}\gamma^5\Psi && \text{(S87)}\\
&= 2im\bar{\Psi}\gamma^5\Psi. && \text{(S88)}
\end{aligned}$$

4. ⋆ Spinors in three dimensions. (5 points)

In this course, we focus on spinors in four dimensions for good reason. In this optional problem, you'll see how the same mathematical structures appear in three dimensions.

a) Consider spinors in three spacetime dimensions. What are the smallest nonzero matrices that can satisfy (6)? Write down three such matrices γ^0, γ^1, and γ^2 explicitly.

Solution: The minimum size is 2×2, and one example set is

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^3 \tag{S89}$$

where the σ^i are the Pauli matrices.

b) We define the spinor Lorentz generators by (8) in any dimension. Since there are now only two spatial dimensions, there is only one rotation generator S^{12}. What phase does a spinor pick up after a 2π rotation?

Solution: We have

$$S^{12} = \frac{i}{4}[i\sigma^1, i\sigma^3] = -\frac{\sigma^2}{2}. \tag{S90}$$

Then a 2π rotation yields

$$\exp\left(-\pi\sigma^2\right) = -I \tag{S91}$$

and therefore a phase of π, just like in 4 dimensions.

c) How does the tensor product of two spinor representations decompose into irreducible representations of the Lorentz group?

Solution: By the exact same logic as in 4 dimensions, we can extract a Lorentz scalar by $\bar{\psi}\psi$, and a Lorentz vector (which only has three components) by $\bar{\psi}\gamma^\mu\psi$. Since spinors have 2 components, the tensor product has 4 components, so this decomposition is complete: there isn't anything else. For instance, you might consider putting $[\gamma^\mu, \gamma^\nu]$ in the middle, but in three dimensions the commutator of two gamma matrices is just another gamma matrix, so you don't get anything new. Similarly, you can't use γ^5 to get anything new, as its analogue here, $i\gamma^0\gamma^1\gamma^2$, is just proportional to the identity.

In two spatial dimensions there are exotic particles called anyons, which can pick up an arbitrary phase after a 2π rotation. However, they can't be described by the conventional quantum fields covered in this course.