PHYS-330: QFT I — Problem Set 7 (due 11/15)

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1. Plane wave solutions of the Dirac equation. (10 points)

In the Weyl representation the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \qquad \sigma^\mu = (\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3). \tag{1}$$

Here, $\mathbb{1}_n$ denotes an $n \times n$ identity matrix.

a) Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 \mathbb{1}_2$.

b) Show that the Dirac equation has positive-frequency plane wave solutions

$$\psi(x) = u_s(p) e^{-ip\cdot x}, \qquad u_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix} \tag{2}$$

where ξ_s is a two-component spinor, i.e. a complex vector with two elements. (Formally, given a matrix A with a complete basis of eigenvectors, with eigenvalues λ_i, we can define a matrix square root $\sqrt{A}$ to have the same eigenvectors, with eigenvalues $\sqrt{\lambda_i}$. But the only thing you need to know to do this problem is $\sqrt{A}\,\sqrt{A} = A$.)

c) Show that if we pick an orthonormal basis of two-component spinors ξ_s with $s \in \{1, 2\}$, satisfying $(\xi_r)^\dagger \xi_s = \delta_{rs}$, then the Dirac spinors satisfy the orthogonality relations

$$u_r(p)^\dagger u_s(p) = 2p_0\, \delta_{rs}, \qquad \bar{u}_r(p) u_s(p) = 2m\, \delta_{rs}. \tag{3}$$

d) Similarly, the Dirac equation has negative-frequency plane wave solutions

$$\psi(x) = v_s(p) e^{ip\cdot x}, \qquad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ -\sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix}. \tag{4}$$

Show that these solve the Dirac equation, and satisfy the orthogonality relations

$$v_r(p)^\dagger v_s(p) = 2p_0\, \delta_{rs}, \qquad \bar{v}_r(p) v_s(p) = -2m\, \delta_{rs}. \tag{5}$$

e) Show the completeness relations for the Dirac spinors,

$$\sum_{s=1}^{2} u_s(p) \bar{u}_s(p) = \not{p} + m\mathbb{1}_4, \qquad \sum_{s=1}^{2} v_s(p) \bar{v}_s(p) = \not{p} - m\mathbb{1}_4. \tag{6}$$

2. Useful spinor identities. (10 points)

a) Prove the Gordon identity,

$$\bar{u}_r(p') \gamma^\mu u_s(p) = \bar{u}_r(p') \left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m} \right) u_s(p) \tag{7}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. (Hint: this can be done using only the Dirac equation and the defining property of the gamma matrices.)

Combinations of gamma matrices can be used to produce a basis Γ^I for the space of 4×4 matrices, where I ranges from 1 to 16. Concretely, we have

$$
\begin{array}{llll}
 & & & \Gamma^{11} = (i/2)[\gamma^0, \gamma^1] \\
 & \Gamma^3 = \gamma^0 & \Gamma^7 = i\gamma^5\gamma^0 & \Gamma^{12} = (i/2)[\gamma^0, \gamma^2] \\
\Gamma^1 = \mathbb{1}_4 & \Gamma^4 = \gamma^1 & \Gamma^8 = i\gamma^5\gamma^1 & \Gamma^{13} = (i/2)[\gamma^0, \gamma^3] \\
\Gamma^2 = \gamma^5 & \Gamma^5 = \gamma^2 & \Gamma^9 = i\gamma^5\gamma^2 & \Gamma^{14} = (i/2)[\gamma^1, \gamma^2] \\
 & \Gamma^6 = \gamma^3 & \Gamma^{10} = i\gamma^5\gamma^3 & \Gamma^{15} = (i/2)[\gamma^1, \gamma^3] \\
 & & & \Gamma^{16} = (i/2)[\gamma^2, \gamma^3]
\end{array}
$$

We define the matrices $\tilde{\Gamma}^I$ similarly, but with lowered Lorentz indices; for example, $\tilde{\Gamma}^4 = \gamma_1 = -\gamma^1 = -\Gamma^4$. (This ensures the equations below won't have annoying extra signs due to the signs in the metric.) An inner product for 4×4 matrices can be defined by

$$\mathrm{tr}(AB) = \sum_{ab} A_{ab} B_{ba}. \tag{8}$$

Under this inner product, the matrices given above are orthogonal, in the sense that

$$\mathrm{tr}(\tilde{\Gamma}^I \Gamma^J) = 4\delta^{IJ} \tag{9}$$

which implies they are linearly independent, and thus indeed form a basis. (You don't have to check (9), but it follows directly from the results you proved in problem set 6.)

b) Show the completeness relation for the Γ matrices,

$$\delta_{ac}\delta_{db} = \frac{1}{4}\sum_I \tilde{\Gamma}^I_{dc}\Gamma^I_{ab}. \tag{10}$$

(Hint: it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd}.)

c) Using (10), show the Fierz identity

$$\tilde{\Gamma}^I_{ab}\Gamma^J_{cd} = \frac{1}{16}\sum_{KL} \mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right] \tilde{\Gamma}^L_{ad}\Gamma^K_{cb}. \tag{11}$$

Contracting both sides of the Fierz identity with four spinors $\bar{u}_{1a}u_{2b}\bar{u}_{3c}u_{4d}$ yields

$$(\bar{u}_1\tilde{\Gamma}^I u_2)(\bar{u}_3\Gamma^J u_4) = \frac{1}{16}\sum_{KL} \mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right] (\bar{u}_1\tilde{\Gamma}^L u_4)(\bar{u}_3\Gamma^K u_2). \tag{12}$$

In other words, the Fierz identity relates the product of a contraction of $\bar{u}_1$ with u_2 and a contraction of $\bar{u}_3$ and u_4 (with arbitrary gamma matrices in the middle) to a combination of contractions of $\bar{u}_1$ with u_4 and $\bar{u}_3$ with u_2. This "Fierz transformation" rearranges how the spinors are contracted with each other, which can be useful in calculations.

d) Find the Fierz transformations for $(\bar{u}_1u_2)(\bar{u}_3u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$ explicitly. (Your final result should only contain the spinors and gamma matrices, not the Γ^I.)

3. Electromagnetism in relativistic notation. (20 points)

In electromagnetism, the Lagrangian is a function of the four-potential $A^\mu = (\phi, \mathbf{A})$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{13}$$

and $J^\mu = (\rho, \mathbf{J})$ is a classical current density.

a) Show that if the current is conserved, $\partial_\mu J^\mu = 0$, then the action remains the same under the gauge symmetry $A_\mu \to A_\mu + \partial_\mu \alpha$, for any smooth function α.

b) Show that the Euler–Lagrange equation for A^μ is

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{14}$$

c) Defining the electric and magnetic fields by $E^i = F^{i0}$ and $\epsilon^{ijk} B^k = -F^{ij}$, show that (14) is equivalent to two of Maxwell's equations.

d) The other two of Maxwell's equations are

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\nu\rho} = 0 \tag{15}$$

which follows directly from the definition of $F_{\mu\nu}$. Interestingly, all four of Maxwell's equations can be written as a single *spinor* equation. Show that the equation

$$\gamma^\nu \gamma^\rho \gamma^\sigma \partial_\nu F_{\rho\sigma} = 2\gamma^\nu J_\nu \tag{16}$$

contains both (14) and (15). (This is just a mathematical trick with no physical meaning, but it's a nice application of the properties of gamma matrices.)

e) In the Lagrangian (13), we could have also included a term of the form $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$, which is Lorentz invariant and gauge invariant. What is it in terms of $\mathbf{E}$ and $\mathbf{B}$? Show that this term is a total derivative, and thus does not contribute to the action.

For the rest of this problem suppose there is no current, $J^\mu = 0$.

f) Construct the stress-energy tensor by directly applying Noether's theorem.

g) The stress-energy tensor you found in part (e) is conserved, but neither symmetric nor gauge invariant. However, we can define an "improved" stress-energy tensor,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho (F^{\mu\rho} A^\nu). \tag{17}$$

Assuming the equations of motion hold, show that $\hat{T}^{\mu\nu}$ is symmetric, gauge invariant, conserved ($\partial_\mu \hat{T}^{\mu\nu} = 0$), and traceless ($\eta_{\mu\nu} \hat{T}^{\mu\nu} = 0$). Furthermore, show that it contains the familiar electromagnetic energy and momentum densities,

$$\hat{T}^{00} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) \,, \qquad \hat{T}^{0i} = (\mathbf{E} \times \mathbf{B})^i. \tag{18}$$

In general, the improved stress-energy tensor considered here is called the Belinfante tensor; it is this stress-energy tensor that sources gravity in general relativity. We didn't see it earlier because the improvement terms are related to the spin angular momentum of the field, which is zero for scalar fields.

4. ⋆ Chern–Simons theory. (5 points)

This optional problem presents a way to produce massive gauge fields.

a) The Proca Lagrangian for a massive vector field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu. \tag{19}$$

Find the equation of motion for A^μ and show that for $m \neq 0$, it implies

$$(\partial^2 + m^2)A^\mu = 0, \qquad \partial_\mu A^\mu = 0. \tag{20}$$

However, this action does not have a gauge symmetry.

b) On the other hand, in two spatial dimensions it is possible to have massive gauge fields. We consider a Lagrangian with a "Chern–Simons" term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho \tag{21}$$

where all indices take the values 0, 1, and 2, and $\epsilon^{012} = 1$. (This new term is a relative of the term we considered in 3(e), but it is not a total derivative.) Show that the action is gauge invariant, and find the Euler–Lagrange equation for A^μ.

c) The Chern–Simons term is sometimes called a "topological" mass term, because it doesn't depend on the metric. Show that the equation of motion implies

$$(\partial^2 + m^2)F^{\mu\nu} = 0 \tag{22}$$

for some m you should find. Thus the field is massive, yet still has a gauge symmetry.

Since there is no Chern–Simons term in three spatial dimensions, it has little role in particle physics, but it is important in the description of topological effects in two-dimensional condensed matter systems.