Stanford University, 2022

1. Plane wave solutions of the Dirac equation. (10 points)

In the Weyl representation the gamma matrices are given by

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}, \qquad \sigma^{\mu} = (\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3), \qquad \bar{\sigma}^{\mu} = (\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3).$$
(1)

Here, $\mathbb{1}_n$ denotes an $n \times n$ identity matrix.

a) Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 \mathbb{1}_2$.

Solution: Inserting the definitions and suppressing the factors of the identity, the expression is

$$(p^{0} - \mathbf{p} \cdot \boldsymbol{\sigma})(p^{0} + \mathbf{p} \cdot \boldsymbol{\sigma}) = (p^{0})^{2} - \frac{1}{2}p^{i}p^{j}\{\sigma^{i}, \sigma^{j}\} = (p^{0})^{2} - |\mathbf{p}|^{2} = p^{2}$$
(S1)

as desired, where we used $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

b) Show that the Dirac equation has positive-frequency plane wave solutions

$$\psi(x) = u_s(p)e^{-ip\cdot x}, \qquad u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix}$$
(2)

where ξ_s is a two-component spinor, i.e. a complex vector with two elements. (Formally, given a matrix A with a complete basis of eigenvectors, with eigenvalues λ_i , we can define a matrix square root \sqrt{A} to have the same eigenvectors, with eigenvalues $\sqrt{\lambda_i}$. But the only thing you need to know to do this problem is $\sqrt{A}\sqrt{A} = A$.)

Solution: By the definition of the matrix square root and the result of part (a), we have

$$\sigma \cdot p = \sqrt{\sigma \cdot p} \cdot \sqrt{\sigma \cdot p}, \quad \bar{\sigma} \cdot p = \sqrt{\bar{\sigma} \cdot p} \cdot \sqrt{\bar{\sigma} \cdot p}, \quad m \mathbb{1}_2 = \sqrt{\sigma \cdot p} \cdot \sqrt{\bar{\sigma} \cdot p} = \sqrt{\bar{\sigma} \cdot p} \cdot \sqrt{\sigma \cdot p}$$
(S2)

where we used $p^2 = m^2$. To show the Dirac equation is satisfied, note that the derivative ∂_{μ} just pulls down a factor of $-ip^{\mu}$. Thus, it suffices to show that

$$(p_{\mu}\gamma^{\mu} - m\mathbb{1}_{4})u_{s}(p) = 0.$$
(S3)

To show this, we just plug in the definitions, giving

$$(p_{\mu}\gamma^{\mu} - m\mathbb{1}_{4})u_{s}(p) = \begin{pmatrix} -m\mathbb{1}_{2} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -m\mathbb{1}_{2} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{s}\\ \sqrt{p\cdot\bar{\sigma}}\,\xi_{s} \end{pmatrix}$$
(S4)

$$= \begin{pmatrix} \sqrt{p \cdot \overline{\sigma}}(-m\mathbb{1}_{2} + \sqrt{p \cdot \overline{\sigma}}\sqrt{p \cdot \overline{\sigma}})\xi_{s} \\ \sqrt{p \cdot \overline{\sigma}}(\sqrt{p \cdot \overline{\sigma}}\sqrt{p \cdot \sigma} - m\mathbb{1}_{2})\xi_{s} \end{pmatrix}$$
(S5)

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} (-m\mathbb{1}_2 + \sqrt{p^2}\mathbb{1}_2)\xi_s \\ \sqrt{p \cdot \overline{\sigma}} (\sqrt{p^2}\mathbb{1}_2 - m\mathbb{1}_2)\xi_s \end{pmatrix}$$
(S6)

$$=0$$
 (S7)

where we used $p^2 = m^2$ in the last step.

c) Show that if we pick an orthonormal basis of two-component spinors ξ_s with $s \in \{1, 2\}$, satisfying $(\xi_r)^{\dagger}\xi_s = \delta_{rs}$, then the Dirac spinors satisfy the orthogonality relations

$$u_r(p)^{\dagger} u_s(p) = 2p_0 \,\delta_{rs}, \qquad \bar{u}_r(p) u_s(p) = 2m \,\delta_{rs}.$$
 (3)

Solution: To take the Hermitian conjugate, note that $(\sigma^{\mu})^{\dagger} = \sigma^{\mu}$. Then we simply have

$$u_{s}^{\dagger}(p)u_{r}(p) = \left(\begin{array}{cc} \xi_{s}^{\dagger}\sqrt{\sigma \cdot p} & \xi_{s}^{\dagger}\sqrt{\overline{\sigma} \cdot p} \end{array}\right) \cdot \left(\begin{array}{c} \sqrt{\sigma \cdot p} \xi_{r} \\ \sqrt{\overline{\sigma} \cdot p} \xi_{r} \end{array}\right)$$
(S8)

$$=\xi_s^{\dagger} \left(p \cdot \sigma + p \cdot \bar{\sigma} \right) \xi_r = 2p^0 \xi_s^{\dagger} \xi_r = 2p^0 \delta_{rs}.$$
(S9)

The proof of the second orthogonality relation is similar:

$$\bar{u}_s(p)u_r(p) = u_s^{\dagger}(p)\gamma^0 u_r(p) = \left(\begin{array}{cc} \xi_s^{\dagger}\sqrt{\bar{\sigma}\cdot p} & \xi_s^{\dagger}\sqrt{\bar{\sigma}\cdot p} \end{array}\right) \cdot \left(\begin{array}{c} \sqrt{\bar{\sigma}\cdot p}\,\xi_r\\ \sqrt{\bar{\sigma}\cdot p}\,\xi_r \end{array}\right) \tag{S10}$$

$$=2\xi_s^{\dagger}\left(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\right)\xi_r=2m\xi_s^{\dagger}\xi_r=2m\delta_{rs}.$$
(S11)

d) Similarly, the Dirac equation has negative-frequency plane wave solutions

$$\psi(x) = v_s(p)e^{ip \cdot x}, \qquad v_s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \,\xi_s \\ -\sqrt{p \cdot \overline{\sigma}} \,\xi_s \end{pmatrix}. \tag{4}$$

Show that these solve the Dirac equation, and satisfy the orthogonality relations

$$v_r(p)^{\dagger} v_s(p) = 2p_0 \,\delta_{rs}, \qquad \bar{v}_r(p) v_s(p) = -2m \,\delta_{rs}.$$
 (5)

Solution: The proof that these plane waves solve the Dirac equation is very similar to that of part (b), except that now the derivative ∂_{μ} pulls down a factor of ip^{μ} , so we want to show

$$(p_{\mu}\gamma^{\mu} + m\mathbb{1}_{4})v_{s}(p) = 0.$$
(S12)

This holds because

$$(p_{\mu}\gamma^{\mu} + m\mathbb{1}_{4})v_{s}(p) = \begin{pmatrix} m\mathbb{1}_{2} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & m\mathbb{1}_{2} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_{s}\\ -\sqrt{p\cdot\bar{\sigma}}\,\xi_{s} \end{pmatrix}$$
(S13)

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} (m \mathbb{1}_2 - \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}}) \xi_s \\ \sqrt{p \cdot \bar{\sigma}} (\sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} - m \mathbb{1}_2) \xi_s \end{pmatrix}$$
(S14)

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} (m \mathbb{1}_2 - \sqrt{p^2} \mathbb{1}_2) \xi_s \\ \sqrt{p \cdot \overline{\sigma}} (\sqrt{p^2} \mathbb{1}_2 - m \mathbb{1}_2) \xi_s \end{pmatrix}$$
(S15)

$$= 0.$$
 (S16)

The proofs of the orthogonality relations are very similar to part (c),

$$v_s^{\dagger}(p)v_r(p) = \left(\begin{array}{cc} \xi_s^{\dagger}\sqrt{\sigma \cdot p} & -\xi_s^{\dagger}\sqrt{\overline{\sigma} \cdot p} \end{array}\right) \cdot \left(\begin{array}{c} \sqrt{\sigma \cdot p}\,\xi_r \\ -\sqrt{\overline{\sigma} \cdot p}\,\xi_r \end{array}\right) \tag{S17}$$

$$=\xi_s^{\dagger}\left(p\cdot\sigma+p\cdot\bar{\sigma}\right)\xi_r=2p^0\xi_s^{\dagger}\xi_r=2p^0\delta_{rs}$$
(S18)

and

$$\bar{v}_s(p)v_r(p) = v_s(p)\gamma^0 v_r(p) = -\left(\begin{array}{cc} \xi_s^{\dagger}\sqrt{\bar{\sigma}\cdot p} & \xi_s^{\dagger}\sqrt{\bar{\sigma}\cdot p} \end{array}\right) \cdot \left(\begin{array}{c} \sqrt{\bar{\sigma}\cdot p}\,\xi_r\\ \sqrt{\bar{\sigma}\cdot p}\,\xi_r \end{array}\right) \tag{S19}$$

$$= -2\xi_s^{\dagger} \left(\sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}}\right) \xi_r = -2m\xi_s^{\dagger} \xi_r = -2m\delta_{rs}.$$
 (S20)

e) Show the completeness relations for the Dirac spinors,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not p + m\mathbb{1}_4, \qquad \sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not p - m\mathbb{1}_4. \tag{6}$$

Solution: This follows straightforwardly from plugging in the definitions,

$$\sum_{s=1}^{2} u_s(p) \bar{u}_s(p) = \sum_{s=1}^{2} \left(\sqrt{p \cdot \sigma} \, \xi_s \right) \left(\xi_s^{\dagger} \sqrt{p \cdot \bar{\sigma}} \quad \xi_s^{\dagger} \sqrt{p \cdot \sigma} \right) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = p_{\mu} \gamma^{\mu} + m \mathbb{1}_4. \tag{S21}$$

$$\sum_{s=1}^{2} v_s(p) \bar{v}_s(p) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p \cdot \sigma} \, \xi_s \\ -\sqrt{p \cdot \bar{\sigma}} \, \xi_s \end{pmatrix} \left(-\xi_s^{\dagger} \sqrt{p \cdot \bar{\sigma}} \quad \xi_s^{\dagger} \sqrt{p \cdot \sigma} \right) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = p_{\mu} \gamma^{\mu} - m \mathbb{1}_4.$$
(S22)

In both cases we used the completeness relation for the two-component spinors, $\sum_{s=1}^{2} \xi_s \xi_s^{\dagger} = \mathbb{1}_2$.

2. Useful spinor identities. (10 points)

a) Prove the Gordon identity,

$$\bar{u}_r(p')\gamma^{\mu}u_s(p) = \bar{u}_r(p')\left(\frac{p'^{\mu} + p^{\mu}}{2m} + \frac{i\sigma^{\mu\nu}(p'_{\nu} - p_{\nu})}{2m}\right)u_s(p)$$
(7)

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]$. (Hint: this can be done using only the Dirac equation and the defining property of the gamma matrices.)

Solution: It's easiest to start from the second term on the right-hand side. Note that

=

$$\bar{u}_r(p')[\gamma^{\mu},\gamma^{\nu}](p'-p)_{\nu}u_s(p) = \bar{u}_r(p')(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})(p'-p)_{\nu}u_s(p)$$
(S23)

$$= \bar{u}_r(p') \left[2(p'+p)^{\mu} - 2(p'\gamma^{\mu} + \gamma^{\mu}p) \right] u_s(p)$$
(S24)

$$= \bar{u}_r(p')(-4m\gamma^{\mu} + 2(p'^{\mu} + p^{\mu}))u_s(p)$$
(S25)

where we used the Dirac equation (S3) in the last step. Plugging this into the right hand side of (7) gives the desired result.

Combinations of gamma matrices can be used to produce a basis Γ^{I} for the space of 4×4 matrices, where I ranges from 1 to 16. Concretely, we have

$$\begin{split} \Gamma^{11} &= (i/2)[\gamma^{0},\gamma^{1}] \\ \Gamma^{3} &= \gamma^{0} & \Gamma^{7} &= i\gamma^{5}\gamma^{0} & \Gamma^{12} &= (i/2)[\gamma^{0},\gamma^{2}] \\ \Gamma^{1} &= \mathbbm{1}_{4} & \Gamma^{4} &= \gamma^{1} & \Gamma^{8} &= i\gamma^{5}\gamma^{1} & \Gamma^{13} &= (i/2)[\gamma^{0},\gamma^{3}] \\ \Gamma^{2} &= \gamma^{5} & \Gamma^{5} &= \gamma^{2} & \Gamma^{9} &= i\gamma^{5}\gamma^{2} & \Gamma^{14} &= (i/2)[\gamma^{1},\gamma^{2}] \\ & \Gamma^{6} &= \gamma^{3} & \Gamma^{10} &= i\gamma^{5}\gamma^{3} & \Gamma^{15} &= (i/2)[\gamma^{1},\gamma^{3}] \\ & & \Gamma^{16} &= (i/2)[\gamma^{2},\gamma^{3}] \end{split}$$

We define the matrices $\tilde{\Gamma}^I$ similarly, but with lowered Lorentz indices; for example, $\tilde{\Gamma}^4 = \gamma_1 = -\gamma^1 = -\Gamma^4$. (This ensures the equations below won't have annoying extra signs due to the signs in the metric.) An inner product for 4×4 matrices can be defined by

$$\operatorname{tr}(AB) = \sum_{ab} A_{ab} B_{ba}.$$
(8)

Under this inner product, the matrices given above are orthogonal, in the sense that

$$\operatorname{tr}(\tilde{\Gamma}^{I}\Gamma^{J}) = 4\delta^{IJ} \tag{9}$$

which implies they are linearly independent, and thus indeed form a basis. (You don't have to check (9), but it follows directly from the results you proved in problem set 6.)

b) Show the completeness relation for the Γ matrices,

$$\delta_{ac}\delta_{db} = \frac{1}{4} \sum_{I} \tilde{\Gamma}^{I}_{dc} \Gamma^{I}_{ab}.$$
 (10)

(Hint: it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd} .)

Solution: We consider a general matrix M, where

$$M_{cd} = \sum_{I} m^{I} \Gamma^{I}_{cd}.$$
 (S26)

Multiplying both sides with M_{cd} , the left-hand side is just M_{ab} , while the right-hand side is

$$\frac{1}{4}\sum_{I}\tilde{\Gamma}^{I}_{dc}\Gamma^{I}_{ab}\sum_{J}m^{J}\Gamma^{J}_{cd} = \sum_{IJ}\Gamma^{I}_{ab}m^{J}\operatorname{tr}(\tilde{\Gamma}^{I}\Gamma^{J}) = \sum_{I}\Gamma^{I}_{ab}m^{I} = M_{ab}.$$
(S27)

Since the two sides match for any matrix M, the completeness relation holds.

c) Using (10), show the Fierz identity

$$\tilde{\Gamma}^{I}_{ab}\Gamma^{J}_{cd} = \frac{1}{16}\sum_{KL} \operatorname{tr}\left[\tilde{\Gamma}^{I}\tilde{\Gamma}^{K}\Gamma^{J}\Gamma^{L}\right]\tilde{\Gamma}^{L}_{ad}\Gamma^{K}_{cb}.$$
(11)

Solution: We note that

$$\tilde{\Gamma}^{I}_{ab}\Gamma^{J}_{cd} = \tilde{\Gamma}^{I}_{ef}\Gamma^{J}_{gh}\delta_{ae}\delta_{fb}\delta_{cg}\delta_{hd}$$
(S28)

$$=\frac{1}{16}\tilde{\Gamma}^{I}_{ef}\Gamma^{J}_{gh}\sum_{KL}\tilde{\Gamma}^{L}_{ad}\Gamma^{L}_{he}\tilde{\Gamma}^{K}_{fg}\Gamma^{K}_{cb}$$
(S29)

$$= \frac{1}{16} \sum_{KL} \operatorname{tr} \left[\tilde{\Gamma}^{I} \tilde{\Gamma}^{K} \Gamma^{J} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \Gamma^{K}_{cb}.$$
(S30)

where we applied the completeness relation twice, and then used the definition of the trace.

Contracting both sides of the Fierz identity with four spinors $\bar{u}_{1a}u_{2b}\bar{u}_{3c}u_{4d}$ yields

$$(\bar{u}_1 \tilde{\Gamma}^I u_2)(\bar{u}_3 \Gamma^J u_4) = \frac{1}{16} \sum_{KL} \operatorname{tr} \left[\tilde{\Gamma}^I \tilde{\Gamma}^K \Gamma^J \Gamma^L \right] (\bar{u}_1 \tilde{\Gamma}^L u_4)(\bar{u}_3 \Gamma^K u_2).$$
(12)

In other words, the Fierz identity relates the product of a contraction of \bar{u}_1 with u_2 and a contraction of \bar{u}_3 and u_4 (with arbitrary gamma matrices in the middle) to a combination of contractions of \bar{u}_1 with u_4 and \bar{u}_3 with u_2 . This "Fierz transformation" rearranges how the spinors are contracted with each other, which can be useful in calculations.

d) Find the Fierz transformations for $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1 \gamma^{\mu} u_2)(\bar{u}_3 \gamma_{\mu} u_4)$ explicitly. (Your final result should only contain the spinors and gamma matrices, not the Γ^I .)

Solution: For the first case, we can apply the Fierz identity with $\tilde{\Gamma}^I = \Gamma^J = \mathbb{1}_4$. Then the right-hand side becomes

$$\frac{1}{16} \sum_{KL} \operatorname{tr} \left[\tilde{\Gamma}^{I} \tilde{\Gamma}^{K} \Gamma^{J} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \Gamma^{K}_{cb} = \frac{1}{16} \sum_{KL} \operatorname{tr} \left[\tilde{\Gamma}^{K} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \Gamma^{K}_{cb}$$
(S31)

$$= \frac{1}{4} \sum_{KL} \delta^{KL} \tilde{\Gamma}^L_{ad} \Gamma^K_{cb}$$
(S32)

$$= \frac{1}{4} \sum_{K} \tilde{\Gamma}_{ad}^{K} \Gamma_{cb}^{K}.$$
 (S33)

Contracting both sides with the spinors, we conclude

$$(\bar{u}_1 u_2)(\bar{u}_3 u_4) = \frac{1}{4} \bigg[(\bar{u}_1 u_4)(\bar{u}_3 u_2) + (\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2) + (\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2)$$
(S34)

$$-(\bar{u}_1\gamma^5\gamma^{\mu}u_4)(\bar{u}_3\gamma^5\gamma_{\mu}u_2) - \frac{1}{8}(\bar{u}_1[\gamma^{\mu},\gamma^{\nu}]u_4)(\bar{u}_3[\gamma_{\mu},\gamma_{\nu}]u_2)\bigg].$$
 (S35)

Note that there is an extra factor of 1/2 in the final term to avoid double counting. This result is admittedly rather complex, but it's indeed rearranged as desired.

For the second case, we take $\tilde{\Gamma}^I = \gamma_{\mu}$ and $\Gamma^J = \gamma^{\mu}$. Computing the right-hand side requires some casework, and use of the trace identities derived in problem set 6. The results are

$$\tilde{\Gamma}^{K} = \mathbb{1}_{4}: \qquad \sum_{L} \operatorname{tr} \left[\gamma_{\mu} \gamma^{\mu} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \delta_{cb} = 16 \delta_{cb} \delta_{ad}, \qquad (S36)$$

$$\tilde{\Gamma}^{K} = \gamma_{\nu}: \qquad \sum_{L} \operatorname{tr} \left[\gamma_{\mu} \gamma_{\nu} \gamma^{\mu} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \gamma^{\nu}_{cb} = -8 \gamma_{\nu ad} \gamma^{\nu}_{cb}, \qquad (S37)$$

$$\tilde{\Gamma}^{K} = \gamma^{5}: \qquad \sum_{L} \operatorname{tr} \left[\gamma_{\mu} \gamma^{5} \gamma^{\mu} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \gamma_{5cb} = -16 \gamma_{5ad} \gamma_{5cb}, \qquad (S38)$$

$$\tilde{\Gamma}^{K} = i\gamma^{5}\gamma_{\nu}: \qquad -\sum_{L} \operatorname{tr}\left[\gamma_{\mu}\gamma^{5}\gamma_{\nu}\gamma^{\mu}\Gamma^{L}\right]\tilde{\Gamma}^{L}_{ad}(\gamma_{5}\gamma^{\nu})_{cb} = 8(i\gamma_{5}\gamma_{\nu})_{ad}(i\gamma_{5}\gamma^{\nu})_{cb}, \qquad (S39)$$

$$\tilde{\Gamma}^{K} = \frac{i}{2} \sum_{L} [\gamma_{\nu}, \gamma_{\rho}]: \qquad \frac{i}{2} \operatorname{tr} \left[\gamma_{\mu} [\gamma_{\nu}, \gamma_{\rho}] \gamma^{\mu} \Gamma^{L} \right] \tilde{\Gamma}^{L}_{ad} \left(\frac{i}{2} [\gamma_{\nu}, \gamma_{\rho}] \right)_{cb} = 0.$$
(S40)

Putting this all together yields

$$(\bar{u}_1\gamma_\mu u_2)(\bar{u}_3\gamma^\mu u_4) = (\bar{u}_1 u_4)(\bar{u}_3 u_2) - \frac{1}{2}(\bar{u}_1\gamma_\mu u_4)(\bar{u}_3\gamma^\mu u_2)$$
(S41)

$$-(\bar{u}_1\gamma^5 u_4)(\bar{u}_3\gamma^5 u_2) - \frac{1}{2}(\bar{u}_1\gamma^5\gamma_\mu u_4)(\bar{u}_3\gamma^5\gamma^\mu u_2).$$
(S42)

3. Electromagnetism in relativistic notation. (20 points)

In electromagnetism, the Lagrangian is a function of the four-potential $A^{\mu} = (\phi, \mathbf{A})$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^{\mu} A_{\mu}, \quad \text{where} \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \quad (13)$$

and $J^{\mu} = (\rho, \mathbf{J})$ is a classical current density.

a) Show that if the current is conserved, ∂_μJ^μ = 0, then the action remains the same under the gauge symmetry A_μ → A_μ + ∂_μα, for any smooth function α.
 Solution: The kinetic term is gauge invariant because F_{μν} is,

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \to \partial_{\mu}A_{\nu} + \partial_{\nu}\partial_{\mu}\alpha - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\alpha = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}.$$
 (S43)

The interaction term is gauge invariant when the current is conserved because

$$J^{\mu}A_{\mu} \to J^{\mu}A_{\mu} + J^{\mu}\partial_{\mu}\alpha = J^{\mu}A_{\mu} + \partial_{\mu}(J^{\mu}\alpha)$$
(S44)

where the change is a total derivative term, which does not affect the action.

b) Show that the Euler-Lagrange equation for A^{μ} is

$$\partial_{\mu}F^{\mu\nu} = J^{\nu}.\tag{14}$$

Solution: The Euler-Lagrange equation is

$$\partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} = \frac{\partial \mathcal{L}}{\partial A_{\nu}}.$$
 (S45)

The right-hand side is clearly $-J^{\nu}$. As for the left-hand side, we use the product rule, giving

$$\partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} = -\frac{1}{2} \partial_{\mu} \left(F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial(\partial_{\mu}A_{\nu})} \right)$$
(S46)

$$= -\frac{1}{2} \partial_{\mu} \left(F^{\rho\sigma} (\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho}) \right)$$
(S47)

$$= -\frac{1}{2}\partial_{\mu} \left(F^{\mu\nu} - F^{\nu\mu}\right)$$
 (S48)

$$= -\partial_{\mu}F^{\mu\nu}.$$
 (S49)

This gives the desired result.

c) Defining the electric and magnetic fields by $E^i = F^{i0}$ and $\epsilon^{ijk}B^k = -F^{ij}$, show that (14) is equivalent to two of Maxwell's equations.

Solution: First set $\nu = 0$. Since $F^{00} = 0$, we have $\partial_i F^{i0} = J^0$, which in vector calculus notation is Gauss's law, $\nabla \cdot \mathbf{E} = \rho$. Next set $\nu = j$. Here we have

$$J^{j} = \partial_0 F^{0j} + \partial_i F^{ij} = -\dot{E}^j - \partial_i \epsilon^{ijk} B^k \tag{S50}$$

which in vector calculus notation is

$$\mathbf{J} = -\dot{\mathbf{E}} + \nabla \times \mathbf{B} \tag{S51}$$

which is Ampere's law.

d) The other two of Maxwell's equations are

$$\epsilon^{\mu\nu\rho\sigma}\partial_{\mu}F_{\nu\rho} = 0 \tag{15}$$

which follows directly from the definition of $F_{\mu\nu}$. Interestingly, all four of Maxwell's equations can be written as a single *spinor* equation. Show that the equation

$$\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\partial_{\nu}F_{\rho\sigma} = 2\gamma^{\nu}J_{\nu} \tag{16}$$

contains both (14) and (15). (This is just a mathematical trick with no physical meaning, but it's a nice application of the properties of gamma matrices.)

Solution: To recover (14), we want to get the current by itself on the right-hand side. Thus, multiply both sides by γ^{μ} and take the trace, giving

$$\operatorname{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma})\partial_{\nu}F_{\rho\sigma} = 2\operatorname{tr}(\gamma^{\mu}\gamma^{\nu})J_{\nu} = 8J^{\mu}.$$
(S52)

Simplifying the left-hand side, we have

$$\operatorname{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma})\partial_{\nu}F_{\rho\sigma} = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\partial_{\nu}F_{\rho\sigma}$$
(S53)

$$=4(-\partial_{\nu}F^{\mu\nu}+\partial_{\nu}F^{\nu\mu})\tag{S54}$$

$$= 8 \,\partial_{\nu} F^{\nu\mu} \tag{S55}$$

where we used the fact that F is antisymmetric, so $\eta^{\rho\sigma}F_{\rho\sigma} = 0$. Equating the two sides and renaming the indices recovers (14).

To recover (15), we want to get a factor of $\epsilon^{\mu\nu\rho\sigma}$ on the left-hand side. Therefore, multiply both sides by $\gamma^5\gamma^{\mu}$ and take the trace. By the results we've derived previously, the right-hand side vanishes and the trace on the left-hand side is proportional to $\epsilon^{\mu\nu\rho\sigma}$, as desired.

e) In the Lagrangian (13), we could have also included a term of the form $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, which is Lorentz invariant and gauge invariant. What is it in terms of **E** and **B**? Show that this term is a total derivative, and thus does not contribute to the action.

Solution: To write this term in terms of E and B, we just expand out the contractions. There are 24 terms, corresponding to the cases where μ , ν , ρ , and σ are all distinct. However, the antisymmetry of F means that groups of eight of them all give the same thing, e.g.

$$\epsilon^{0123}F_{01}F_{23} = \epsilon^{1023}F_{10}F_{23} = \epsilon^{0132}F_{01}F_{32} = \epsilon^{1032}F_{10}F_{32}$$
$$= \epsilon^{2301}F_{23}F_{01} = \epsilon^{2310}F_{23}F_{10} = \epsilon^{3201}F_{32}F_{01} = \epsilon^{3210}F_{32}F_{10}.$$
(S56)

There are thus only three distinct terms to write down. Adjusting the index position for convenience,

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 8(\epsilon_{0123}F^{01}F^{23} + \epsilon_{0213}F^{02}F^{13} + \epsilon_{0312}F^{03}F^{12})$$

$$= 8(-F^{01}F^{23} + F^{02}F^{13} - F^{03}F^{12})$$
(S57)
(S58)

$$= 8(F^{10}F^{23} + F^{20}F^{31} + F^{30}F^{12})$$
(S59)
$$= 8(F^{10}F^{23} + F^{20}F^{31} + F^{30}F^{12})$$

$$= 8(F^{10}F^{23} + F^{20}F^{31} + F^{30}F^{12})$$
(S60)
(S60)

$$= -8 \mathbf{E} \cdot \mathbf{B}. \tag{S61}$$

Now, to show the term is a total derivative, we note that

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\left[\partial_{\mu}A_{\nu}\partial_{\rho}A_{\sigma} - \partial_{\nu}A_{\mu}\partial_{\rho}A_{\sigma} - \partial_{\mu}A_{\nu}\partial_{\sigma}A_{\rho} + \partial_{\nu}A_{\mu}\partial_{\sigma}A_{\rho}\right]$$
(S62)

$$=4\epsilon^{\mu\nu\rho\sigma}\partial_{\mu}A_{\nu}\partial_{\rho}A_{\sigma} \tag{S63}$$

$$=4\epsilon^{\mu\nu\rho\sigma}\partial_{\mu}(A_{\nu}\partial_{\rho}A_{\sigma})-4\epsilon^{\mu\nu\rho\sigma}A_{\nu}\partial_{\mu}(\partial_{\rho}A_{\sigma})$$
(S64)

$$=\partial_{\mu}\left(4\epsilon^{\mu\nu\rho\sigma}A_{\nu}\partial_{\rho}A_{\sigma}\right)\tag{S65}$$

which is a total derivative as desired. Note that the second and third steps in this derivation just follow from the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

For the rest of this problem suppose there is no current, $J^{\mu} = 0$.

f) Construct the stress-energy tensor by directly applying Noether's theorem.Solution: Applying Noether's theorem as usual, we have

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}A_{\rho})}\partial^{\nu}A_{\rho} - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^{\nu}A_{\rho} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.$$
 (S66)

g) The stress-energy tensor you found in part (e) is conserved, but neither symmetric nor gauge invariant. However, we can define an "improved" stress-energy tensor,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_{\rho} (F^{\mu\rho} A^{\nu}).$$
(17)

Assuming the equations of motion hold, show that $\hat{T}^{\mu\nu}$ is symmetric, gauge invariant, conserved $(\partial_{\mu}\hat{T}^{\mu\nu} = 0)$, and traceless $(\eta_{\mu\nu}\hat{T}^{\mu\nu} = 0)$. Furthermore, show that it contains the familiar electromagnetic energy and momentum densities,

$$\hat{T}^{00} = \frac{1}{2} (|\mathbf{E}|^2 + |\mathbf{B}|^2), \qquad \hat{T}^{0i} = (\mathbf{E} \times \mathbf{B})^i.$$
 (18)

Solution: Adding on the new term gives

$$\hat{T}^{\mu\nu} = (\partial_{\rho}F^{\mu\rho})A^{\nu} + F^{\mu\rho}F_{\rho}^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.$$
(S67)

When the equations of motion hold, $\partial_{\rho}F^{\mu\rho} = 0$, so we may simply drop the first term, giving

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$
(S68)

where we cleaned up the indices a bit. In this form, the stress-energy tensor is clearly symmetric, and it is gauge invariant because it is written in terms of the gauge invariant F alone. To show conservation, it is easiest to note that

$$\partial_{\mu}\hat{T}^{\mu\nu} = \partial_{\mu}T^{\mu\nu} + \partial_{\mu}\partial_{\rho}(F^{\mu\rho}A^{\nu}) = 0$$
(S69)

where the first term vanishes by Noether's theorem, and the second term vanishes by the antisymmetry of F. Finally, to show tracelessness, we note that

$$\eta_{\mu\nu}\hat{T}^{\mu\nu} = -\eta_{\mu\nu}\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = -F_{\nu\sigma}F^{\nu\sigma} + F_{\rho\sigma}F^{\rho\sigma} = 0.$$
 (S70)

Now let's consider the components. For the energy density, we have

$$\hat{T}^{00} = -\eta_{\rho\sigma} F^{0\rho} F^{0\sigma} + \frac{1}{4} \eta^{00} F_{\rho\sigma} F^{\rho\sigma}$$
(S71)

$$=F^{01}F^{01} + F^{02}F^{02} + F^{03}F^{03} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}$$
(S72)

$$= |\mathbf{E}|^{2} + \frac{1}{2}(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{23}F^{23} + F_{31}F^{31})$$
(S73)

$$= |\mathbf{E}|^2 + \frac{1}{2}(-|\mathbf{E}|^2 + |\mathbf{B}|^2)$$
(S74)

$$=\frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) \tag{S75}$$

as desired. For the momentum density, we have

$$\hat{T}^{0i} = -\eta_{\rho\sigma} F^{0\rho} F^{i\sigma} = F^{0j} F^{ij} = E^j \epsilon^{ijk} B^k = (\mathbf{E} \times \mathbf{B})^i.$$
(S76)

In general, the improved stress-energy tensor considered here is called the Belinfante tensor; it is this stress-energy tensor that sources gravity in general relativity. We didn't see it earlier because the improvement terms are related to the spin angular momentum of the field, which is zero for scalar fields.

4. \star Chern–Simons theory. (5 points)

This optional problem presents a way to produce massive gauge fields.

a) The Proca Lagrangian for a massive vector field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_{\mu}A^{\mu}.$$
(19)

Find the equation of motion for A^{μ} and show that for $m \neq 0$, it implies

$$(\partial^2 + m^2)A^{\mu} = 0, \qquad \partial_{\mu}A^{\mu} = 0.$$
 (20)

However, this action does not have a gauge symmetry.

Solution: The derivation of the equation of motion is similar to problem 3(b), but now a new term appears on the right-hand side, giving

$$-\partial_{\mu}F^{\mu\nu} = m^2 A^{\nu}. \tag{S77}$$

This is the equation of motion. Applying ∂_{ν} to both sides yields

$$m^2 \partial_\nu A^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0 \tag{S78}$$

by the antisymmetry of F, which shows $\partial_{\mu}A^{\mu} = 0$ when m is nonzero. Plugging this result back into the original equation of motion gives

$$-\partial_{\mu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) = -\partial^{2}A^{\nu} = m^{2}A^{\nu}$$
(S79)

as desired.

b) On the other hand, in two spatial dimensions it is possible to have massive gauge fields. We consider a Lagrangian with a "Chern–Simons" term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha}{4}\epsilon^{\mu\nu\rho}F_{\mu\nu}A_{\rho}$$
(21)

where all indices take the values 0, 1, and 2, and $\epsilon^{012} = 1$. (This new term is a relative of the term we considered in 3(e), but it is not a total derivative.) Show that the action is gauge invariant, and find the Euler-Lagrange equation for A^{μ} .

Solution: The proof that the action is gauge invariant is similar to 3(e). Note that the change in the Lagrangian under a gauge transformation is

$$\delta \mathcal{L} = \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} \partial_{\rho} \alpha = \frac{\alpha}{2} \epsilon^{\mu\nu\rho} (\partial_{\mu} A_{\nu}) \partial_{\rho} \alpha = \partial_{\rho} \left(\frac{\alpha}{2} \epsilon^{\mu\nu\rho} \alpha \partial_{\mu} A_{\nu} \right)$$
(S80)

which is a total derivative, as desired. As for the Euler–Lagrange equation, starting from our solution to 3(b), we pick up an extra term on both the left-hand and right-hand sides, giving

$$-\partial_{\mu}F^{\mu\nu} + \frac{\alpha}{2}\epsilon^{\mu\nu\rho}\partial_{\mu}A_{\rho} = \frac{\alpha}{4}\epsilon^{\mu\rho\nu}F_{\mu\rho}.$$
(S81)

We can clean this up using the antisymmetry of ϵ to give

$$\partial_{\mu}F^{\mu\nu} = \frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho}.$$
(S82)

c) The Chern–Simons term is sometimes called a "topological" mass term, because it doesn't depend on the metric. Show that the equation of motion implies

$$(\partial^2 + m^2)F^{\mu\nu} = 0$$
 (22)

for some m you should find. Thus the field is massive, yet still has a gauge symmetry.

Solution: This is a bit fiddly. The idea is that we want to use the equation of motion twice. But in its current form, we can't really do much to it, because of the factor of ϵ on the right-hand side which already contracts with most of the indices. However, we can clear factors of ϵ out of the way by contracting with additional copies of ϵ , and using

$$\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^{\nu}_{\alpha}\delta^{\rho}_{\beta} - \delta^{\nu}_{\beta}\delta^{\rho}_{\alpha}.$$
(S83)

To get started, we contract both sides of (S82) with $\epsilon_{\nu\alpha\beta}$, giving

$$\epsilon_{\nu\alpha\beta}\partial_{\mu}F^{\mu\nu} = \alpha F_{\beta\alpha}.\tag{S84}$$

Now we can apply ∂^{β} to both sides and use the equation of motion again, giving

$$\epsilon_{\nu\alpha\beta}\partial^{\beta}\partial_{\mu}F^{\mu\nu} = \frac{\alpha^2}{2}\epsilon_{\mu\alpha\nu}F^{\mu\nu}.$$
(S85)

To get rid of the ϵ 's again, we contract both sides with $\epsilon^{\alpha\delta\sigma}$, giving

$$\partial^{\delta}\partial_{\mu}F^{\mu\sigma} - \partial^{\sigma}\partial_{\mu}F^{\mu\delta} = \alpha^2 F^{\sigma\delta}.$$
(S86)

Finally, to get the factor of ∂^2 we want, we can apply ∂_δ to both sides, giving

$$\partial^2 \partial_\mu F^{\mu\sigma} = \alpha^2 \partial_\delta F^{\sigma\delta} \tag{S87}$$

where a term vanished due to the antisymmetry of F. Cleaning this up, we have

$$0 = (\partial^2 + \alpha^2)(\partial_{\mu}F^{\mu\nu}) = (\partial^2 + \alpha^2)\left(\frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho}\right)$$
(S88)

where we used the equation of motion a final time. Now we just clear away the factor of ϵ by contracting with $\epsilon_{\nu\alpha\beta}$, which finally gives the result with $m = \alpha$.

Since there is no Chern–Simons term in three spatial dimensions, it has little role in particle physics, but it is important in the description of topological effects in two-dimensional condensed matter systems.