PHYS-330: QFT I Problem Set 8 (due 11/29)

Stanford University, 2022 Bernhard Mistlberger, Kevin Zhou

1. Magnetic moments in quantum electrodynamics. (15 points)

The interaction Hamiltonian in quantum electrodynamics for a Dirac field of charge q is

$$H_I = q \int d^3x \, \bar{\Psi} \gamma^\mu \Psi A_\mu. \tag{1}$$

To understand the physical meaning of this expression, we can evaluate its matrix elements in states $|p, s\rangle = \sqrt{2E_p}\, a_s^\dagger(p)|0\rangle$ with a single fermion. We assume the electromagnetic field is in a quantum state with negligible field uncertainty, so that its field operator A_μ can be replaced with a time-independent classical expectation value $A_\mu^{\text{cl}}(\mathbf{x})$.

a) Show that the matrix elements of the Schrodinger picture Hamiltonian are

$$\langle p, s|H_I|p', s'\rangle = q \int d^3x \, e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\gamma^\mu u_{s'}(p') A_\mu^{\text{cl}}(\mathbf{x}) \tag{2}$$

when $|p', s'\rangle \neq |p, s\rangle$.

In nonrelativistic quantum mechanics, the single-particle states are $|\mathbf{p}, s\rangle$. If the particle has charge q' and g-factor g, then its magnetic moment is $\mu = (gq'/2m)\,\mathbf{S}$ where $\mathbf{S}$ is the spin. In terms of the particle's position $\mathbf{x}$, the Hamiltonian contains the terms

$$H_I^{\text{nr}} = q' A^0(\mathbf{x}) - \boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{x}). \tag{3}$$

To find the values of q' and g, we equate matrix elements in the nonrelativistic limit,

$$\langle p, s|H_I|p', s'\rangle = 2m\langle \mathbf{p}, s|H_I^{\text{nr}}|\mathbf{p}', s'\rangle \text{ when } |\mathbf{p}|, |\mathbf{p}'| \ll m \tag{4}$$

where the factor of $2m$ converts between relativistic and nonrelativistic normalization. We could evaluate both sides for general $A_\mu^{\text{cl}}(\mathbf{x})$, but it is easier to consider two special cases. In both cases it will be helpful to use the Gordon identity from set 7.

b) Evaluate both sides of (4) in a static electric field, corresponding to general $A^0(\mathbf{x})$ and $\mathbf{A} = 0$. Show that they agree when $q' = q$, as one would expect.

c) Show that if the fermion was a classical spinning ball with uniform mass and charge density, then its g-factor would be 1. This strongly disagrees with the measured value.

d) Evaluate both sides of (4) in a static magnetic field, corresponding to $A^0 = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}(\mathbf{x})$, and infer the value of g. (Hint: to relate spin in the Dirac theory to spin in the nonrelativistic theory, recall that in the nonrelativistic theory, spinors ξ_s have two components and the spin operator is $\mathbf{S} = \boldsymbol{\sigma}/2$. You already showed in problem set 7 how the Dirac spinors $u_s(p)$ are built from ξ_s. You will have to integrate by parts, so assume $\mathbf{A}$ and $\mathbf{B}$ vanish at infinity.)

The agreement of this value you found in part (d) with the experimentally measured value for the electron was one of the early triumphs of the Dirac equation.

2. Decays of the Higgs boson. (10 points)

The Standard Model contains three charged leptons, the electron e, muon μ, and tau τ, which are described by Dirac fields and differ only by their mass. It also contains a spinless particle called the Higgs boson, described by a real scalar field h. The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) - \frac{1}{2} m_h^2 h^2 + \sum_i \bar{\Psi}_i (i\partial\!\!\!/ - m_i)\Psi_i \tag{5}$$

where $i \in \{e, \mu, \tau\}$, and the numeric values of the masses are

$$m_h = 125\,\mathrm{GeV}, \qquad m_e = 511\,\mathrm{keV}, \qquad m_\mu = 105.7\,\mathrm{MeV}, \qquad m_\tau = 1777\,\mathrm{MeV}. \tag{6}$$

The Higgs field couples to the charged leptons by a Yukawa coupling proportional to mass,

$$\mathcal{L}_{\text{int}} = -\sum_i \frac{m_i}{v} h \bar{\Psi}_i \Psi_i \tag{7}$$

Here, v is a constant associated with the breaking of electroweak symmetry, but for this problem you will only need its value, $v = 246\,\mathrm{GeV}$. For this problem, all of your final answers should be numeric, and given to at least two significant figures.

a) Compute the partial decay rate for a Higgs boson to an electron-positron pair $\Gamma_{H\to e^+e^-}$ at leading non-vanishing order in perturbation theory, giving your answer in eV. (Hint: sum over final spin states, and reuse results from problem set 5.)

b) Find the ratios $\Gamma_{H\to\mu^+\mu^-}/\Gamma_{H\to e^+e^-}$ and $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to e^+e^-}$.

c) Find the probability a Higgs boson decays to $\tau^+\tau^-$, also known as the branching ratio

$$\mathrm{BR}_{H\to\tau^+\tau^-} = \frac{\Gamma_{H\to\tau^+\tau^-}}{\sum_X \Gamma_{H\to X}}. \tag{8}$$

The denominator is the total decay rate of the Higgs boson, equal to 4.1 MeV. (To check your answer, you can consult the so-called Yellow Report.)

The decay of Higgs bosons to taus was confirmed experimentally only recently [1,2], and first evidence for the Higgs coupling to muons has been detected [3,4]. Establishing the coupling of the Higgs to electrons remains a monumental experimental challenge.

3. Electron-positron annihilation to muons. (15 points)

One of the key successes of quantum electrodynamics is its description of particle creation and annihilation processes at relativistic energies. In this exercise we will consider the process $e^+e^- \to \mu^+\mu^-$, where the electron and muon are Dirac fields of charge e, and mass m_e and m_μ respectively.

a) Let the initial momenta be $p^\mu_{e^+}$ and $p^\mu_{e^-}$ and the final momenta be $p^\mu_{\mu^+}$ and $p^\mu_{\mu^-}$. Find the scattering matrix element $\mathcal{M}$ for this process, to leading nonvanishing order in e.

b) Let $|\bar{\mathcal{M}}|^2$ be the square of the matrix element, summed over final spin states and averaged over initial spin states. Compute $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, and the Mandelstam variables s and t, where

$$s = (p_{e^+} + p_{e^-})^2, \quad t = (p_{e^-} - p_{\mu^-})^2, \quad u = (p_{e^-} - p_{\mu^+})^2. \tag{9}$$

For the rest of this problem, we specialize to the center of mass frame.

c) Rewrite $|\bar{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, s, and the angle θ between $\mathbf{p}_{e^-}$ and $\mathbf{p}_{\mu^-}$.

d) Starting from equation (4.84) of Peskin and Schroeder, compute the differential cross section $d\sigma/d\Omega$ and the total cross section.

You can see section 5.1 of Peskin and Schroeder to get started or check your answer. But note that the book neglects the mass of the electron, while here we account for it.

4. $\star$ Nonminimal couplings. (5 points)

In problem 1, you found the charge and g-factor for a Dirac fermion minimally coupled to the electromagnetic field, i.e. via the simplest possible interaction (1). However, if the fermion is composite, or interacts with heavier particles, we might need additional terms to describe the coupling. For all parts of this problem, you should adopt the formalism of problem 1 and work in the nonrelativistic limit. Detailed calculations are not needed; qualitative final answers (with justification) are sufficient.

a) The g-factors of the proton and neutron are *not* given by the result you found in problem 1. When the Dirac equation was invented, physicists explained this by adding an additional term to $\mathcal{H}_I$, proportional to $i\bar{\Psi}[\gamma^\mu, \gamma^\nu]\Psi\, F_{\mu\nu}$. Show that the physical effect of such a term is indeed to shift the magnetic dipole moment.

b) What is the physical effect of a term proportional to $\bar{\Psi}[\gamma^\mu, \gamma^\nu]\gamma^5\Psi\, F_{\mu\nu}$?

The above two are the only terms with dimension 5, and are therefore the simplest non-minimal couplings one could consider. Next, let's consider some dimension 6 terms.

c) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\Psi\, \partial^\nu F_{\mu\nu}$?

d) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\gamma^5\Psi\, \partial^\nu F_{\mu\nu}$?