PHYS-330: QFT I — Problem Set 9 (due 12/8)

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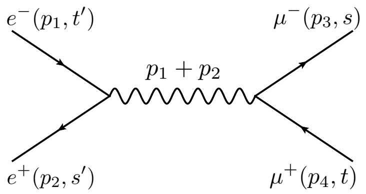
This problem set guides you through the calculation of loop diagrams and their application to the renormalization of QED. Unlike previous problem sets, it is ordered sequentially: each problem is easiest to approach after completing the previous problems. For reference, you may find it useful to consult sections 6.3 and 7.5 of Peskin and Schroeder, but this problem set is self-contained and its results are somewhat more general.

We will use dimensional regularization with $d = 4 - 2\epsilon$. To avoid confusion, we will rename the $i\epsilon$ in the Feynman propagator to $i0$ in this problem set.

1. $e^+e^- \to \mu^+\mu^-$ at one loop. (8 points)

In the previous problem set, you considered the leading order (or "tree level") contribution to the scattering matrix element for $e^+e^- \to \mu^+\mu^-$,

$\mathcal{M}^{\mathrm{LO}} =$



(1)

$$= \bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2 + i0}\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1) \tag{2}$$

which is proportional to e^2. The next-to-leading order contribution $\mathcal{M}^{\mathrm{NLO}}$ is of order e^4, and contains Feynman diagrams with one loop.

a) Draw all Feynman diagrams that contribute to $\mathcal{M}^{\mathrm{NLO}}$.

b) Write down $\mathcal{M}^{\mathrm{NLO}}$ using the Feynman rules. You don't have to evaluate the loop integrals; this is just to show you examples of their typical form.

2. Feynman parameters. (12 points)

As you saw in problem 1, loop integrals take a few generic forms. For simplicity, let's consider a scalar theory, which doesn't have nontrivial factors in the numerator of the loop integrand. Then some generic loop integrals are the "tadpole", "bubble", and "triangle",

$$I_T(a; m^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2 - m^2)^a} \tag{3}$$

$$I_B(a, b; p^2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p)^2)^b} \tag{4}$$

$$I_\triangle(a, b, c; p_1, p_2) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a((k+p_1)^2)^b((k-p_2)^2)^c} \tag{5}$$

named after the kinds of Feynman diagrams in which they appear. Here we have suppressed $i0$ terms, but you should keep in mind that they're implicitly there, which will be important in the last subpart. We regard a, b, and c as general real exponents, though in practice they will usually be positive integers. The dimension d is also a real parameter.

To compute the bubble and triangle integrals, it is helpful to introduce Feynman parameters, which combine the factors into the denominator into a power of a single quantity, like the tadpole integral. We will first have to build up some mathematical machinery.

a) The gamma function is defined by

$$\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x} \tag{6}$$

and for positive integer n, satisfies $\Gamma(n) = (n-1)!$. Show that for real ν,

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dx\, x^{\nu-1} e^{-xA}. \tag{7}$$

b) Show that for general ν_i,

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{8}$$

Here, the x_i are called Feynman parameters. (Hint: you should *not* base your answer on the derivation in Peskin and Schroeder, which only works for integer ν. Instead, start from the result you proved in part (a).)

c) Apply (8) to $I_B(a, b; p^2)$ to get an integrand whose denominator contains a single quantity raised to the $a + b$ power.

d) Complete the evaluation of $I_B(a, b; p^2)$ by Wick rotating to Euclidean signature and performing all the remaining integrals. As a hint, the Euclidean integration measure is $d^d k_E = |k_E|^{d-1} d|k_E|\, d\Omega_d$, where the surface area of a unit sphere in d dimensions is

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{9}$$

As a second hint, you can use the result

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx\, x^{a-1}(1-x)^{b-1}. \tag{10}$$

To check your answer, if you set $d = 4 - 2\epsilon$ then you should find

$$I_B(1, 1; p^2) = \left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} \frac{i\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{(4\pi)^2 (1 - 2\epsilon)\epsilon\Gamma(1 - 2\epsilon)}. \tag{11}$$

3. Passarino–Veltman reduction. (10 points)

More generally, loop integrands will have momenta and other factors in the numerator. However, we can often use symmetry properties and a technique called Passarino–Veltman reduction to reduce them to the simpler loop integrands considered in problem 2. For example, consider the rank 2 tadpole integral

$$I_T^{\mu\nu}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^a}. \tag{12}$$

Because the right-hand side is a Lorentz tensor, the left-hand side must be as well. As there are no momenta in the integral, the only option is the metric, so we must have

$$I_T^{\mu\nu}(a; m^2) = \eta^{\mu\nu} A \tag{13}$$

for some Lorentz scalar A. Contracting both sides with the metric, we find

$$A = \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - m^2)^a} \tag{14}$$

$$= \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{(k^2 - m^2) + m^2}{(k^2 - m^2)^a}. \tag{15}$$

This is just two "scalar" tadpole integrals, as defined in (3), so we conclude

$$I_T^{\mu\nu}(a; m^2) = \frac{\eta^{\mu\nu}}{d} \left[I_T(a-1; m^2) + m^2 I_T(a; m^2) \right]. \tag{16}$$

a) Apply the same reasoning to the rank 3 and rank 4 tadpole integrals,

$$I_T^{\mu\nu\rho}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho}{(k^2 - m^2)^a}, \tag{17}$$

$$I_T^{\mu\nu\rho\sigma}(a; m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2 - m^2)^a}. \tag{18}$$

b) Similarly, the rank 1 and 2 bubble integrals can be written as

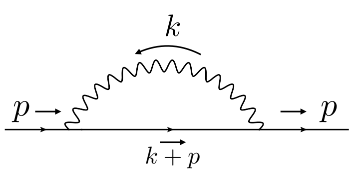
$$I_B^{\mu}(a, b; p^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^a ((k+p)^2)^b} = p^\mu C.$$

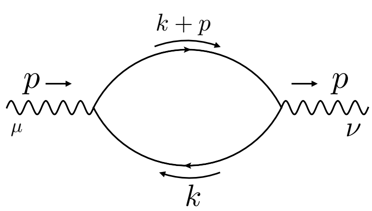
$$I_B^{\mu\nu}(a, b; p^2) = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2)^a ((k+p)^2)^b} = \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) A_\perp + \frac{p^\mu p^\nu}{p^2} B_{||}. \tag{19}$$

Express $A_\perp$ and $B_{||}$ and C in terms of p^2 and the scalar bubble integral (4).

4. Self-energy corrections in QED. (10 points)

Loop corrections to the two-point correlation function play a key role in renormalization, and are often referred to as self-energy corrections. The one-loop self-energy corrections for the electron and photon in QED are given by the diagrams below.

$\Sigma(p^2) =$ 

$\Pi^{\mu\nu}(p^2) =$ 

(20)

Throughout this problem we will work in massless QED, $m_e = 0$.

a) Using the Feynman rules, write down $\Sigma(p^2)$ and $\Pi^{\mu\nu}(p^2)$. Since we are viewing these quantities as corrections to the electron and photon propagators, you should *not* include factors for the external legs, such as $u_s(p)$ or ϵ_μ, to get a scalar matrix element. Instead, Σ should be a 4×4 spinor matrix and $\Pi^{\mu\nu}$ a rank 2 Lorentz tensor.

b) Evaluate the resulting loop integrals, expressing your final result in terms of p^2 and ϵ. (Hint: once you handle the gamma matrices, all of the integrals you get will be ones evaluated earlier in the problem set.)

c) The Euler–Mascheroni constant is defined as

$$\gamma_E = -\int_0^\infty dx\, e^{-x} \log x \approx 0.577. \tag{21}$$

Show that

$$\Gamma(1-\epsilon) = 1 + \gamma_E \epsilon + O(\epsilon^2). \tag{22}$$

d) Using (22), expand your results from part (b), dropping terms that vanish as $\epsilon \to 0$.

5. ⋆ The scalar triangle integral. (5 points)

In this optional problem, we consider a somewhat more difficult loop integral.

a) Evaluate the scalar triangle integral $I_\triangle(a, b, c; p_1, p_2)$ when $p_1^2 = p_2^2 = 0$, but for general a, b, c, and d, and give your answer in terms of $s = (p_1 + p_2)^2$.

b) It turns out there is a simple relation between the triangle and bubble integrals,

$$I_\triangle(1, 1, 1; p_1, p_2)\bigg|_{p_1^2 = p_2^2 = 0} = C I_B(1, 1; (p_1 + p_2)^2). \tag{23}$$

Find the coefficient C in terms of d and s.