PHYS-330: QFT I — Section 1 (9/29)

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1. Tensor survival skills.

A key prerequisite for quantum field theory is comfort manipulating tensors. In previous courses, you should have seen the four-vectors for momentum, potential, and current

$$p^\mu = (E, \mathbf{p}), \quad A^\mu = (\phi, \mathbf{A}), \quad J^\mu = (\rho, \mathbf{J}). \tag{1}$$

In this course, the metric tensor $\eta_{\mu\nu}$ has elements

$$\eta_{00} = 1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -1. \tag{2}$$

There is also an inverse metric tensor $\eta^{\mu\nu}$, which has elements

$$\eta^{00} = 1, \quad \eta^{11} = \eta^{22} = \eta^{33} = -1. \tag{3}$$

Indices can be raised and lowered using the metric or inverse metric, e.g.

$$p_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} p^\nu \tag{4}$$

which implies

$$p_0 = p^0, \quad p_1 = -p^1, \quad p_2 = -p^2, \quad p_3 = -p^3 \tag{5}$$

Here, the sum over ν is called a contraction. Contractions always involve one upper and one lower spacetime index, and throughout this course we'll use the Einstein summation convention, i.e. we won't explicitly write the summation sign for contractions. Note that while the *position* of a spacetime index is very important (p^μ is not the same thing as p_μ), names of contracted indices are arbitrary ($\eta_{\mu\nu} p^\nu$ means exactly the same thing as $\eta_{\mu\rho} p^\rho$).

a) Show that $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$, where

$$\delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

This shows that δ is the identity tensor. (It is also sometimes written as η^μ_ρ.)

Solution: Note that $\eta^{\mu\nu}$ is only nonzero when $\mu = \nu$, and likewise $\eta_{\nu\rho}$ is only nonzero when $\nu = \rho$. Thus, when $\mu \neq \rho$, the contraction has to be zero. When $\mu = \rho = 0$, the contraction gives $1^2 = 1$, while when $\mu = \rho \neq 0$, the contraction gives $(-1)^2 = 1$, establishing the result.

b) Find the numeric value of $\eta^{\mu\nu}\eta_{\mu\nu}$.

Solution: Summing over μ and ν, the only terms that contribute are those with $\mu = \nu$, giving

$$\eta_{00}\eta^{00} + \eta_{11}\eta^{11} + \eta_{22}\eta^{22} + \eta_{33}\eta^{33} = 4. \tag{1}$$

c) A simple example of a covector (an object with one lower index) is the four-derivative

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{7}$$

Explicitly write out $\partial \cdot A = \partial_\mu A^\mu$ and $p \cdot A = p_\mu A^\mu$ in terms of components.

Solution: Note the difference in signs:

$$\partial \cdot A = \partial_t \phi + \partial_x A^x + \partial_y A^y + \partial_z A^z, \qquad p \cdot A = E\phi - p^x A^x - p^y A^y - p^z A^z. \tag{2}$$

The metric tensor is special because it is the unique 2-index ("rank 2") tensor whose elements stay the same under Lorentz transformations. There is only one other tensor with these properties, the rank 4 Levi–Civita tensor ϵ, whose components are

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} \pm 1 & \mu, \nu, \rho, \sigma \text{ all distinct} \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

Here, $\epsilon^{0123} = 1$ and the Levi–Civita tensor is totally antisymmetric, meaning that the values of its components flips sign whenever two indices are exchanged, e.g. $\epsilon^{2103} = -1$.

d) Find the numeric values of ϵ^{3210} and ϵ_{0123}.

Solution: The first requires two exchanges, so it is $+1$. The second is

$$\eta_{0\mu}\eta_{1\nu}\eta_{2\rho}\eta_{3\sigma}\epsilon^{0123} = \eta_{00}\eta_{11}\eta_{22}\eta_{33}\epsilon^{0123} = (-1)^3 = -1. \tag{3}$$

e) Show that $g_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}$ is zero for any ρ and σ.

Solution: A contraction of an antisymmetric and symmetric pair of indices always vanishes, as

$$g_{\mu\nu}\epsilon^{\mu\nu\rho\sigma} = -g_{\nu\mu}\epsilon^{\nu\mu\rho\sigma} = -g_{\mu\nu}\epsilon^{\mu\nu\rho\sigma} \tag{4}$$

where in the first step we used symmetry of g and antisymmetry of ϵ, and in the second step we exchanged the names of the indices μ and ν. The only quantity which is equal to its own negative is zero.

f) It turns out that

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\delta\gamma} = c_1(\delta^\delta_\rho \delta^\gamma_\sigma - \delta^\gamma_\rho \delta^\delta_\sigma), \tag{9}$$
$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\delta} = c_2\, \delta^\delta_\sigma, \tag{10}$$
$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = c_3. \tag{11}$$

You don't have to prove these expressions, though it's worth thinking about why they must be true. Using any method, find the numeric values of c_1, c_2, and c_3.

Solution: To prove the first identity, let's take the concrete case $\rho = \delta = 2$ and $\sigma = \gamma = 3$, where

$$\epsilon_{\mu\nu 23}\epsilon^{\mu\nu 23} = c_1. \tag{5}$$

The terms that contribute are $(\mu, \nu) = (0, 1)$ and $(\nu, \mu) = (1, 0)$. They're equal, and the first one is $\epsilon_{0123}\epsilon^{0123} = -1$, so we conclude $c_1 = -2$.
To get c_2, we simply contract δ and ρ on both sides of the first identity. The result is

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\gamma} = -2(\delta^\rho_\rho \delta^\gamma_\sigma - \delta^\gamma_\rho \delta^\rho_\sigma) = -2(4\delta^\gamma_\sigma - \delta^\gamma_\sigma) = -6\delta^\gamma_\sigma \tag{6}$$

so that $c_2 = -6$. Finally, contracting σ and γ here gives $c_3 = -24$.
Alternatively, we could have started with $c_3 = -24$ by counting permutations, then reverse engineered c_2 and c_1 by running the above reasoning in reverse.

This problem contains all the tensor manipulation skills you'll need for the entire course. By the way, the fact that the metric tensor and Levi–Civita tensor are the only two "invariant tensors" has a geometric meaning. Contracting the metric tensor with two vectors gives their spacetime inner product, while contracting the Levi–Civita tensor with four vectors gives the signed spacetime volume of the parallelepiped they span.

2. Using Lorentz symmetry.

The fact that the metric and the Levi–Civita tensor are the only invariant tensors severely constrains the forms that results can take in relativistic theories.

For example, suppose the answer to a physical question is a four-vector, but doesn't depend on any physical quantities besides Lorentz scalars (e.g. particle masses). Then the answer has to be built out of $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ alone, but there's no way to contract copies of them to get a four-vector, so the answer must be zero. On the other hand, if the answer is a tensor with 2 upper indices, the only possible answer is $f\,\eta^{\mu\nu}$ where f is a Lorentz scalar. (We showed previously that you can't get anything else by using ϵ.)

a) What is the most general form if the answer is a tensor with 3 upper indices?

Solution: Because only pairs of indices can be contracted, any answer has to have an even number of indices, so the answer here has to be zero.

b) What if the answer is a tensor with 4 upper indices?

Solution: The general form is

$$f_1\eta^{\mu\nu}\eta^{\rho\sigma} + f_2\eta^{\mu\rho}\eta^{\nu\sigma} + f_3\eta^{\mu\sigma}\eta^{\nu\rho} + f_4\epsilon^{\mu\nu\rho\sigma}. \tag{7}$$

Now suppose the question involves some four-vector, such as the four-momentum p^μ of a particle. In this case, p^μ can show up in the answer, which permits more general forms. If the answer is a four-vector, it can be $f\,p^\mu$, and if the answer is a tensor with two upper indices, it can be

$$f_1\,p^\mu p^\nu + f_2\,\eta^{\mu\nu} \tag{12}$$

where f_1 and f_2 are Lorentz scalars. This really is the general answer: it already includes any more complicated expression you can write. For example,

$$\begin{aligned} p^\mu p^\nu p^\rho p^\sigma \eta_{\rho\delta}\eta^{\delta\gamma}\eta_{\gamma\sigma} &= p^\mu p^\nu p^\rho p^\sigma \delta^\gamma_\rho \eta_{\gamma\sigma} && (13)\\ &= p^\mu p^\nu p^\rho p^\sigma \eta_{\rho\sigma} && (14)\\ &= (p\cdot p)\,p^\mu p^\nu && (15) \end{aligned}$$

which corresponds to $f_1 = p\cdot p$ and $f_2 = 0$. And there's no term involving ϵ, because contracting two copies of p with it yields zero.

c) What is the most general form if the answer is a tensor with 3 upper indices, which can involve p^μ?

Solution: To do this systematically, consider casework. With zero copies of p, we get nothing. With one copy, we have two or four indices left over, which has to be an η or ϵ. With two copies, we again get nothing, and with three copies we have one term, giving

$$f_1\eta^{\mu\nu}p^\rho + f_2\eta^{\mu\rho}p^\nu + f_3\eta^{\nu\rho}p^\mu + f_4\epsilon^{\mu\nu\rho\sigma}p_\sigma + f_5 p^\mu p^\nu p^\rho. \tag{8}$$

Be careful to avoid writing terms that are always zero, or equivalent to other terms. Once you get the hang of this, you'll be able to check answers at a glance by looking at their tensor structure, just as you've learned to check answers by dimensional analysis.